Complete Markets Strikes Back: Revisiting Risk Sharing Tests under Preference Heterogeneity*

Gang Sun†
University of Oslo
JOB MARKET PAPER

March 28, 2011

Abstract

Previous risk sharing tests reject the hypothesis of complete markets because in the data the consumption dispersion increases over the life cycle and individual consumption co-moves with income. In this paper I show that, if there is positive correlation between heterogeneous discount factors and heterogeneous income growth rates, all these tests arrive at the conclusion of imperfect risk sharing even if the markets are actually complete. I first demonstrate that it is not only theoretically possible, but also quantitatively admissible for a simple complete market model to account for the empirical evidence on both the consumption dispersion and the co-movement of consumption and income. Then, I evaluate the complete market (CM) model and the standard incomplete market (SIM) models in a calibrated framework with heterogeneous preferences and heterogeneous income profiles. I find that the CM model outperforms the SIM models in matching the data.

JEL Classfication: E21 D31 D58 D91

Keywords: complete markets, incomplete markets, heterogeneity, consumption, inequality, risk, insurance

*I am deeply indebted to Kjetil Storesletten, my PhD thesis supervisor, for his encouragement and suggestions. I also thank Kaiji Chen, Juan Carlos Conesa, Fatih Guvenen, Jonathan Heathcote, Greg Kaplan, Tim Kehoe, Finn Kydland, Hamish Low, Ellen McGrattan, Franck Portier, Ed Prescott, Vincenzo Quadrini and all participants of seminars in the XIV Workshop on Dynamic Macroeconomics at Vigo, the 5th NHH-UiO Workshop, the Ragnar Frisch Centre for Economic Research, the University of Oslo, ETH-Zurich, for helpful comments. I thank the Norwegian Research Council for its financial support for my one year visiting in the Federal Reserve Bank of Minneapolis, where part of this paper were written. All the errors are my own.

†Department of Economics, University of Oslo, Box 1095, Blindern, 0371 Oslo, Norway. Email: gangs@econ.uio.no

1 Introduction

How well do people insure against income risks? The answer of this question is important for redistribution policies and macroeconomic modelling. Recent risk sharing tests exploit the information from the micro data sets of both consumption and income distributions. One type of tests is to study the change of consumption *dispersion*: Deaton and Paxson (1994) study the consumption dispersion for each birth-cohort and find that the consumption dispersion increases significantly over the life cycle. As they point out, the fanning out over the life cycle of both consumption and income distributions indicates that, under the standard separable preferences, the individual's labor income must include highly persistent and uninsurable risk. The other type of tests, which is theoretically more transparent while empirically more demanding, uses the *co-movement* of consumption and income. Attanasio and Davis (1996) study the relative movement of consumption of cohort-education groups as a response to a relative wage change. Blundell, Preston and Pistaferri (2008, BPP hereafter) generalize this idea and estimate the pass-through of individual income shocks to consumption growth using a constructed panel of both consumption and income. These authors find that individual or group consumption co-moves significantly with the income[1].

All of the above risk sharing tests share the same conclusion: the hypothesis of complete markets is soundly rejected. In a standard complete market (CM) model, there would be neither significant dispersion of consumption over the life cycle nor significant co-movement of individual or group consumption and income.

The striking evidence from these tests has a profound impact on the agenda of quantitative economists. Many dynamic quantitative model builders turn to another workhorse model: the class of standard incomplete market (SIM) models, as the SIM models outperform the CM models in all these tests. For example, Storesletten et al. (2004) find that the SIM model with highly persistent labor market shocks can account for the empirical increase of consumption dispersion over the life cycle in Deaton and Paxson (1994), whose estimates are based on the U.S. data of Consumption Expenditure Suvery (CEX) before early 1990s. To account for the recent empirical evidence from Heathcote et al. (2010a), who use the CEX waves from 1980 to 2006, Heathcote et al. (2010b) introduce additional channels of insurance, such as labor supply, education choice and family formation, into the SIM model; their model can match

[1] The predecessors are Cochrane (1991), who rejects the hypothesis of complete markets for long illness and involuntary job loss, and Mace (1991), who rejects the complete market hypothesis with power utility.

the rise of the consumption dispersion both over time and over the life cycle. Moreover, in Sun (2010) and Kaplan and Violante (2010), they find that the pass-through of individual income shocks to consumption growth in the SIM models is closer to the empirical estimate of BPP (2008) than its CM counterpart, although SIM models generate much less consumption insurance than that in the data and a counter-factual age trend of the degree of risk sharing.

In this paper, I revisit the implication of these risk sharing tests and re-evaluate the CM and SIM models, both qualitatively and quantitatively. I find that not only is the hypothesis of complete markets *not* rejected, but also it outperforms SIM in the sense that the CM model can match more closely the observed consumption dispersion and the co-movement of consumption and income.

To revive the hypothesis of complete markets, I extend an otherwise standard CM model to allow for positive correlation between heterogeneous discount factors and heterogeneous income growth rates. Why does it work? The intuition is simple: with discount rate heterogeneity, people with different discount factors have different profiles of consumption path. Even if markets are complete and consumption paths are deterministic, consumption dispersion will eventually increase over the life cycle when their consumption paths diverge. As an individual's consumption growth rate is increasing in her discount factor when markets are complete, consumption will co-move with income if the discount factor is positively correlated with the income growth rate. As a result, with preference heterogeneity, the risk sharing tests arrive at the conclusion of imperfect risk sharing even if the markets are actually complete.

To deliver the key messages, I will first discuss the methodology of and evidence from the risk sharing tests. Then, I present two classes of complete market models. One with stochastic income and one with predictable income change. Both are standard models except that the agents' time preference is no longer restricted to be homogeneous. Simple as these models are, they deliver the key insight about how the presence of discount rate heterogeneity may change the age profile of consumption distribution and how its interaction with income profile heterogeneity may affect the co-movement of consumption and income. I demonstrate that it is not only theoretically possible, but also quantitatively admissible for the CM model to simultaneously account for both the observed increase of consumption dispersion over the life cycle and the co-movement of consumption and income. To further evaluate the CM and SIM models by their performances in matching the data, I move to the quantitative exercise using a calibrated framework which nests both SIM and CM. For the measure of co-movement, I take the approach developed independently both in my companion paper Sun (2010) and Kaplan and Violante (2010) and estimate BPP's coefficients using the simulated consumption and income data from the model.

The first new ingredient in my model is the discount rate heterogeneity, which is not directly observable. Nevertheless, there are some indirect evidence on the dispersion of discount factors by experiments designed to reveal individual's preference (Barsky et al. 1997), by Euler equation estimation (Lawrance 1991, Alan and Browning 2010), and by analyzing the difference of retirement wealth with the same life-time income (Hendricks 2007b). After the seminal paper by Krusell and Smith (1998), it is well-known in the macroeconomic literature that discount rate heterogeneity can potentially play an important role in accounting for wealth distribution. Hendricks (2007a) follows this insight and estimates the distribution of discount factors in a calibrated life cycle model by matching the observed wealth inequality. In this paper, I follow Hendricks (2007a) and use the wealth Gini coefficient as the calibration target for the distribution of discount factors.

The second new ingredient is the Heterogeneous Income Profiles (HIP). This view of income process is emphasized by some authors (e.g. Lillard and Weiss 1979, Hause 1980 and Guvenen 2007, 2009), which is an alternative to the view of Restricted Income Profiles (RIP) with very persistent income shocks[2]. Without additional information or ad hoc model specification, it is difficult for the econometricians to tell apart the predictable income change, such as the heterogeneous income growth, from the unpredictable income change, such as the permanent income shocks. In a recent paper by Primiceri and Rens (2009), they estimate a HIP model and find that a significant part of income change is predictable. In my model, I will take a more general view of the HIP process, in which the RIP process is a special case.

The third new ingredient is the positive correlation between individual discount factors and individual income growth rates. I provide two justifications: a model of education and a health model where the discount factor incorporates the survival rate. Because both discount factors and income growth rates are unobservable, at least in the usual data set, I will choose this parameter to match the empirical co-movement of consumption and income.

This paper contributes to the large literature of the risk-sharing tests during the past 20 years (e.g. Cochrane 1991, Mace 1991, Altonji, Hayashi and Kotlikoff 1992, Townsend 1994, Deaton and Paxson 1994, Attanasio and Davis 1996, Blundell and Preston 1998, BPP 2008, Heathcote et al. 2009). In particular, it contributes to a recent new strand of research which casts doubt on the risking sharing tests with the presence of preference heterogeneity (Dubois 2001, Mazzocco and Saini 2009, and Schulhofer-Wohl 2010). Different from this paper, these authors focus on the implication of the heterogeneity in individual risk aversion. In the

[2] See Guvenen (2009) for a summary.

complete market, the consumption paths of the less risk-averse agents are steeper because they would have lower consumption when aggregate resource is high and higher consumption when aggregate resource is low. If the less risk-averse agents choose the idiosyncratic income process with higher variance, then there would be positive co-movement between consumption and income.

Several papers have addressed the importance of preference on the consumption dispersion in a complete market setup. Deaton and Paxson (1994) mention that the nonseparable preferences between consumption and leisure could possibly generate an increase of consumption dispersion over the life cycle, even the markets are complete. Storesletten et al. (2001) take this theoretical possibility seriously and study a calibrated complete market model with a CES utility function of consumption and leisure. They find that, however, in order to generate the increase of consumption dispersion over the life cycle, it implies a counter-factual age profile of hours inequality. Different from their model, the increase of consumption dispersion in my model does not come from the assumption of non-separable preferences. Badel and Huggett (2010) study in a complete market setup how the preference shocks can account for the life-cycle profile of both consumption and hours dispersion. Their focus is the role of preference shifters, not time preferences as in this paper.

The rest of the paper is organized as follows: Section 2 presents the empirical risk sharing tests, their methodology and results. Section 3 presents two simple versions of complete market (CM) models with discount rate heterogeneity and heterogeneous income profiles, derives analytical results of the consumption dispersion and the co-movement of consumption and income, matches the data by a simple quantitative excercise and presents justifications for the positive correlation between individual's discount factor and income growth rate. Section 4 sets up a class of large scale models, which nests the standard incomplete market (SIM) models and the CM model, and calibrates the parameters. Section 5 brings the calibrated model to the data and evaluates the CM and SIM models by the consumption dispersion and the co-movement of consumption and income. Section 6 concludes. Proofs and robustness check are in the Appendix.

2 The Risk Sharing Tests

2.1 Methodology

The degree of risking sharing can be affected by the nature of market structure and/or the nature of income risks. To test risking sharing, a growing literature uses the information from the micro data sets of consumption and income distributions. In terms of the structure of the data they use, these tests can be classified as two categories: One is to use the data sets of consumption and income distributions *separately*, study the change of consumption distribution over the life cycle (Deaton and Paxson 1994) and compare it with the change of income distribution accordingly. The other is to exploit the *joint-distribution* of consumption and income. To do this, one has to construct either a synthetic panel (Attanasio and Davis 1996) or a combined panel data of consumption and income (BPP 2008). These two lines of risk sharing tests share the same methodology which can be illustrated by the following example:

Let us look at the textbook version of Permanent Income /Life Cycle Hypothesis (PILCH). Consider an individual saving problem where she is endowed with a stochastic income process y_{it}, lives for T periods, and only has access to a risk-free bond with net interest rate r. Assume further that the period utility is linear quadratic with the discount factor $\beta = 1/(1+r)$.

Assume the borrowing constraints are loose enough to make the first order conditions hold. Solving the model analytically yields that the consumption follows a martingale process. And the change of consumption is given by

$$\pi \Delta c_{i,t} = \frac{r}{1+r} \sum_{s=0}^{T-t} \frac{(E_t - E_{t-1}) y_{i,t+s}}{(1+r)^s}, \tag{1}$$

where $\pi \equiv 1 - \frac{1}{(1+r)^{T-t+1}}$. The right hand side is the annuity value of the innovations to future income between $t-1$ and t.

In the data, we observe the change of income distribution, both over time and over the life cycle. We can single out the idiosyncratic part of income, which is usually identified as "shocks" from the eyes of the econometricians. At this moment, I do not take any stand on the true empirical income process, but instead consider three extreme cases of the composition of the idiosyncratic income $y_{i,t}$.

Case A: Pure unit root income shock: $y_{i,t} = z_{i,t}$ and $z_{i,t} = z_{i,t-1} + \eta_{i,t}$, where $\eta_{i,t}$ is the permanent shock.

In this case, $\Delta y_{it} = \eta_{i,t}$. We can solve analytically to get $\Delta c_{i,t} = \eta_{i,t}$. The innovation of income passes one-to-one to the change of consumption. No risk sharing is attained for the permanent shocks.

Case B: Pure predictable and heterogeneous income profile: $y_{i,t} = \theta_i t$, where θ_i is the slope of individual income path.

Since all the income changes are predictable, the period by period risk-free bonds essentially complete the markets. Thus we have $\Delta y_{i,t} = \theta_i$ and $\Delta c_{i,t} = 0$. The individual consumption does not respond to income at all. Perfect risk sharing is attained.

Case C: Pure I.I.D. income shock: $y_{i,t} = \varepsilon_{i,t}$, where ε_{it} is the transitory shock.

If shocks are transitory, we have $\Delta y_{i,t} = \varepsilon_{i,t} - \varepsilon_{i,t-1}$ and $\Delta c_{i,t} = \frac{r}{1+r}\pi^{-1}\varepsilon_{i,t}$. Notice that $\pi \approx 1$ if $T - t$ is a large number and $\frac{r}{1+r} \approx r$ if r is a small number. Therefore, $\Delta c_t^i \approx r\varepsilon_{i,t}$ with large $T - t$ and small r. There is some pass-through from income to consumption, but the pass-through is small. A large degree of risk sharing of transitory shocks is attained.

The above three different income processes imply completely different consumption responses. By looking at the change of consumption distribution and income distribution, or look at their co-movement if possible, we can identify the nature of market structure and/or the nature of the income risks, and hence we can test the competing hypotheses of risk sharing.

2.2 Evidence

2.2.1 Consumption dispersion

Deaton and Paxson (1994) first construct the age profile of the variance of log consumption using Consumer Expenditure Survey (CEX) data before early 1990s. They find that the consumption dispersion of each cohort increases over the life cycle by 0.28 log points. Using longer time span from 1980 to 2006, Heathcote et al. (2010a) find that the log variance of consumption increases over the life cycle by 0.057 when controlling for year effects and by 0.13 when controlling for cohort effects. Their estimates are plotted in Figure 1. While Heathcote et al. (2010a) do not take a stand on which empirical strategy is better, Heathcote et al. (2005) suggest controlling for year effects[3].

[Insert Figure 1]

The consensus is: consumption dispersion increases significantly over the life cycle. From the previous PILCH example under standard seperable preferences, this result suggests that the individual's labor income must include highly persistent and uninsurable risks.

2.2.2 Co-movement of consumption and income

Between group inequality Obviously, the joint distribution of consumption and income contains more information than separate data sets of consumption and income distributions. The reason why most researchers focus on the cross-sectional consumption distribution is mainly due to the lack of high quality longitude data of both consumption and income. To

[3] Recently, several authors argue that the diary survey in CEX is better designed than the interview survey. Attanasio et al. (2007) use the diary survey of the CEX and Attanasio et al. (2010) combine these two surveys. They find that the increase of consumption inequality from 1980 to 2006 rises twice as much as the result from Heathcote et al. (2010), who use the interview survey. But they did not report the increase of consumption dispersion over the life cycle.

circumvent this data deficiency problem, Attanasio and Davis (1996) study the effect of relative wage movements among birth cohorts and education groups on the consumption distribution. They use cross-sectional consumption and income data sets to construct a synthetic panel of both consumption and income. They find that the relative wage movements among cohort-education groups of men drove large changes in the distribution of household consumption during 1980s, which they conclude as "strikingly sharp rejections of the consumption insurance hypothesis".

Pass-through: BPP To generalize Attanasio and Davis (1996)'s idea, BPP (2008) denote partial insurance as the degree of the transmission of income shocks to consumption growth. They assume that unexplained log income can be decomposed into a unit root permanent part and an i.i.d. transitory part; they also assume the log consumption as

$$\Delta \log c_{i,t} = \phi_{i,t}\eta_{i,t} + \psi_{i,t}\varepsilon_{i,t} + u_{i,t}, \tag{2}$$

where $\eta_{i,t}$ is the innovation of the permanent shock and $\varepsilon_{i,t}$ is the transitory shock, $u_{i,t}$ is the error term. Since $\phi_{i,t}$ is the pass-through of permanent income shocks to consumption change, it is a natural measure of (the lack of) consumption insurance.

BPP map food data into expenditure data using the estimates of a demand function for food that are present in both the Panel Study of Income Dynamics (PSID) and the CEX to create an unbalanced panel data series of consumption and income. BPP's main finding is that, in the whole sample, the estimate of ϕ and ψ is 0.6423 and 0.0533, respectively. In other words, a 10 percent permanent (transitory) shock of disposable income induces a 6.4(0.5) percent change in household's nondurable consumption. In the complete market, $\phi = 0$, and therefore their "partial insurance" result rejects the hypothesis of complete markets.

Sun (2010) and Kaplan and Violante (2010) measure consumption insurance in a class of simulated life-cycle incomplete market models using BPP's method. In the simulated model, since shocks are known, it is straight forward to compute $\widehat{\phi}_t$ and $\widehat{\psi}_t$ from the definition equation (2) by standard OLS approach. To confront the model with data, however, we have to apply BPP's estimation by instruments, since shocks are unknown to the econometrician:

$$\widehat{\phi}_t^{Instr} = \frac{E(\Delta\widehat{c}_t(\Delta\widehat{y}_{t-1} + \Delta\widehat{y}_t + \Delta\widehat{y}_{t+1}))}{E(\Delta\widehat{y}_t(\Delta\widehat{y}_{t-1} + \Delta\widehat{y}_t + \Delta\widehat{y}_{t+1}))}, \tag{3}$$

$$\widehat{\psi}_t^{Instr} = -\frac{E(\Delta\widehat{c}_t\Delta\widehat{y}_{t+1})}{E(\Delta\widehat{y}_t\Delta\widehat{y}_{t+1})}, \tag{4}$$

where $\Delta\widehat{c}_t$ and $\Delta\widehat{y}_t$ denotes $\log c_t$ and $\log y_t$, respectively. Although the direct OLS and the instrument method are in principle different, Sun (2010) proves that both the instrument and

OLS estimators are consistent estimators of ϕ and ψ, if the consumption model (2) is not mis-specified.

2.2.3 Message from the tests

The message from all the above risk sharing tests is: because in the data we all find that consumption distribution co-moves with income distribution, the hypothesis of perfect risk sharing is rejected. In other words, either (1) the hypothesis of complete markets must be rejected or (2) the hypothesis of fully predictable heterogeneous income profiles must rejected, or both. Notice that (2) is a special case of (1).

This message, however, is premature, if there exists preference heterogeneity. In the rest of the paper, I will revisit the implication of these tests by extending the standard complete and incomplete market models to include preference heterogeneity.

3 Complete Market: simple models

In this section, I will present two classes of complete market models. One with stochastic income and one with predictable income change. Both are standard models except that the agents' time preferences are no longer restricted to be homogeneous. Simple as the models are, they deliver the key insight on how the presence of discount rate heterogeneity may change the age profile of consumption distribution and how its interaction with income profile heterogeneity may affect the co-movement of consumption and income.

3.1 Environment

Demography Consider an economy populated by T overlapping-generations, each of which consists of I agents. Each individual enters the labor market at age 1 and can live a maximum of T periods. Agents face mortality risks. The probability of surviving between age $t-1$ and age t is denoted by ξ_t, with $\xi_1 = 1$ and $\xi_{T+1} = 0$. Define $\varsigma_t \equiv \prod_{\tau=1}^{t} \xi_\tau$ as the unconditional survival rate.

Technology For any $t \geq 1$, there is a stochastic event $s_t \in S$. Let $s^t \equiv [s_1, s_2, ..., s_t]$ denote the history of events up and until time t. Each agent is endowed with a sequence of consumption good $y_i(s^t)$.

Individual log income process is restricted to the following form:

$$\begin{aligned} \log y_{i,t} &= \log w + \alpha_i + \underbrace{\theta_i t + z_{i,t} + \varepsilon_{i,t}}_{\text{idiosyncratic income change}} , \\ z_{i,t} &= \rho z_{i,t-1} + \eta_{i,t} , \end{aligned} \tag{5}$$

where $\rho \in [0,1]$, $\eta_{i,t} \sim N(0,\sigma_\eta^2)$, $\varepsilon_{i,h} \sim N(0,\sigma_\varepsilon^2)$, $z_{i,1} = 0$. w is the marginal return to aggregate labor supply and α_i is the fixed effect which is predetermined before the agent enters the labor market. The next three terms captures the idiosyncratic income change. θ_i is the heterogeneity in the growth rate of individual income. The idiosyncratic shocks part consists of a permanent (or AR(1) when $\rho < 1$) part z_t^i and a transitory (i.i.d.) part ε_t^i. Define $\sigma_\theta \equiv var(\theta_i) \geq 0$. This income process nests both the Heterogeneous Income Profiles (HIP) process where $\sigma_\theta > 0$ and the Restricted Income Profiles (RIP) process where $\sigma_\theta = 0$.

It can be the case that $\eta_{i,t}$ and $\varepsilon_{i,t}$ are "shocks" to the econometricians, but predictable to the individual i. In one of the following models, I will allow $\eta_{i,t}$ and $\varepsilon_{i,t}$ to be predictable to the agent i.

There is no aggregate uncertainty. For the analysis of allocations in the stationary equilibrium, we do not need to specify the production function and the supply of capital, as long as we are given the equilibrium marginal product of capital net of depreciation r.

Preference Agent i's preference over the stream of consumption is given by

$$u_i(c) = E\sum_{t=1}^{T} \varsigma_t \beta_i^{t-1} \frac{[c_i(s^t)]^{1-\gamma}}{(1-\gamma)},$$

where β_i is the individual's time discount factor, γ is the relative risk aversion. Note that β_i is individual specific. When each individual i is born, her discount factor β_i is drawn from a distribution F_β with mean μ_β and variance $\sigma_\beta^2 \geq 0$. This nests the standard model where the discount rate heterogeneity is absent and $\sigma_\beta = 0$.

Then I will discuss models with different market structures and/or income shocks.

3.2 Complete markets with stochastic income

Suppose the markets are complete. The allocation of this economy can be characterized by the solution to the social planner's problem of each birth-cohort, with Pareto weight λ_i for each agent i:

$$\max \sum_i \lambda_i E \sum_{t=1}^{T} \varsigma_t \beta_i^{t-1} \frac{c_i(s^t)^{1-\gamma}}{1-\gamma} \tag{6}$$

$$s.t. \sum_{t,s^t} \sum_i \frac{c_{i,t}(s^t) - y_{i,t}(s^t)}{(1+r)^{t-1}} \leq 0,$$

Combining the first order conditions for any two agent i and j, we have

$$\frac{c_{i,t}(s^t)}{c_{j,t}(s^t)} = (\frac{\lambda_i}{\lambda_j})^{\frac{1}{\gamma}} (\frac{\beta_i}{\beta_j})^{\frac{t-1}{\gamma}} \tag{7}$$

This simple necessary condition tells us two important messages: 1. The ratio of consumption of any two individuals does not depend on their own income history, therefore each consumer will get full insurance under this market structure. 2. If $\beta_i = \beta_j$, the consumption ratio is a constant which is determined at the beginning of the life cycle. Testing constant relative marginal utility of consumption is the key logic of the test of complete markets. If $\beta_i \neq \beta_j$, however, the ratio of c_i and c_j will change expotentially with age. For given $\{\lambda_i\}_{i=1}^{I}$, the log consumption can be solved analytically as

$$\log c_{i,t} = \frac{t-1}{\gamma} \log \beta_i + \log \lambda_i^{1/\gamma} + \Gamma_t, \tag{8}$$

where $\Gamma_t \equiv \log \frac{(1+r)^{\frac{t}{\gamma}-1} \sum_{t,s^t} \sum_i \frac{y_{i,t}(s^t)}{(1+r)^{t-1}}}{\sum_{t=1}^{T} [(1+r)^{\frac{(1-\gamma)t}{\gamma}} \sum_i \lambda_i^{1/\gamma} \beta_i^{(t-1)/\gamma}]}$. Notice that we can now write $c_{i,t}$ instead of $c_i(s_t)$, since consumption at time t is not a function of the realization of s^t. This equation tells us that the agent with higher β has a steeper path of log consumption and the agent with higher Pareto weight is given a higher initial consumption level.

3.2.1 Consumption dispersion

The last term in equation (8) is not individual specific. So, only the variance of the first two terms can be positive. The variance of log consumption is:

$$var \log c_{i,t} = \frac{(t-1)^2}{\gamma^2} \sigma^2_{\log \beta} + \frac{2(t-1)}{\gamma} cov(\log \beta_i, \log \lambda_i^{1/\gamma}) + var(\log \lambda_i^{1/\gamma}), \tag{9}$$

where $\sigma^2_{\log \beta} \equiv var(\log \beta_i)$.

Notice the last term is the initial consumption dispersion. Therefore, the variance of log consumption is a quadratic function of time t. Its shape in domain $[1, T]$ depends on $cov(\log \beta_i, \lambda_i^{1/\gamma})$. If $cov(\log \beta_i, \lambda_i^{1/\gamma})$ is not a big negative number, the variance of log consumption will eventually increase quadraticly in the life cycle. If $cov(\log \beta_i, \log \lambda_i^{1/\gamma})$ is positive, the life cycle profile will be increasing from the begining of the life cycle, otherwise, it is U-shaped. Formally, we have the following proposition:

Proposition 1 *(1) If $T > t^* \equiv 1 - \gamma cov(\log \beta_i, \log \lambda_i^{1/\gamma}) / \sigma^2_{\log \beta}$, then for $t > t^*$, the variance of log consumption in the complete market model increases convexly with age t. If $cov(\log \beta_i, \log \lambda_i^{1/\gamma}) < 0$, then the consumption dispersion age profile is U-shaped.*

(2) If $cov(\log \beta_i, \log \lambda_i^{1/\gamma}) > \frac{-(T-1)\sigma^2_{\log \beta}}{2\gamma}$, then the consumption dispersion increases over the life cycle. Therefore, the hypothesis of complete markets cannot be rejected by the observed increase of consumption dispersion over the life cycle.

(3) Consumption dispersion over the life cycle increases with $cov(\beta_i, \theta_i)$.

Proof. See Appendix. ∎

This proposition says the dispersion of discount factors can be gradually transmited into the dispersion of consumption over the life cycle. It should be noticed that the increase of consumption dispersion in this complete market model does not come from the assumption of non-separable preferences between consumption and leisure, as in Storesletten et al. (2001).

The last claim comes from the fact that the second term in (9) is a linear function of t and the Pareto weight λ_i is an increasing function of one's life time income. Although we theoretically we do not need $cov(\beta_i, \theta_i) > 0$ to generate an increase of $var(\log c_{i,t})$, a positive correlation between β_i and θ_i will generate a larger increase of consumption dispersion.

3.2.2 Co-movement of consumption and income

When markets are complete, individual's consumption should not respond to income since there are no uninsurable income risks, even if there exists discount rate heterogeneity. In the OLS estimation by BPP's measure in the model, all the permanent and transitory pass-throughs should be zero. However, the econometricians do not know individual's true income process. The best thing they can do is to use the panel data of consumption and income. If they observe one's income and one's consumption are correlated, they conclude that there is uninsurable income "risk".

To see this, we derive the increase of log consumption from $t-1$ to t,

$$\Delta_t \log c_{it} = \frac{\log \beta_i}{\gamma} - \Delta_t \Gamma_t. \tag{10}$$

This equation shows that the individual consumption growth is an increasing function of her discount factor. If individual's income growth is also correlated with her discount factor, then BPP's measure by the instrument estimation is no longer zero. This would be true if the individual income process follows heterogeneous income profile and the covariance between individual's income growth rate θ_i and discount factor β_i is positive. Formally, we have the following proposition:

Proposition 2 *(1) If markets are complete, the pass-through of permanent shocks $\phi_{t,CM}$ and transitory shocks $\psi_{t,CM}$ are zero for any t.*

(2) The BPP's estimators by instrument are $\widehat{\phi}_{t,CM} = \frac{cov(\log \beta_i, \theta_i)}{\gamma\{\sigma_\theta^2 + \rho[(\rho-1)(\rho^3-1)\sum_{\tau=1}^{t-2}\rho^{2(\tau-1)} + \rho(\rho-1)+1]\sigma_\eta^2/3\}}$ *and* $\widehat{\psi}_{t,CM} = \frac{cov(\log \beta_i, \theta_i)}{\gamma\{\sigma_\varepsilon^2 - \sigma_\theta^2 - (\rho-1)[\rho(\rho-1)\sum_{\tau=1}^{t-1}\rho^{2\tau}+1]\sigma_\eta^2\}}$.

(3) If $cov(\beta_i, \theta_i) > 0$, then $\widehat{\phi}_{t,CM}$ is positive. Therefore, the hypothesis of complete markets cannot be rejected by the observed positive $\widehat{\phi}_{t,CM}$ from BPP's test.

Proof. *See Appendix.* ■

This proposition says that the covariance of discount factor and income growth rate can be transmitted into the co-movement of consumption and income. Therefore, it is not informative to test full risk-sharing by the co-movement of consumption and income.

When $\rho < 1$ or $\sigma_\theta^2 > 0$, BPP's coefficients are no longer consistent estimates of ϕ and ψ because of model mis-specification. Two polar cases are worth noticing:

When $\rho = 1$,

$$\widehat{\phi}_{t,CM} = \frac{cov(\theta_i, \widetilde{\beta}_i)}{\gamma(\sigma_\theta^2 + \sigma_\eta^2/3)}; \widehat{\psi}_{t,CM} = \frac{cov(\theta_i, \widetilde{\beta}_i)}{\gamma(\sigma_\varepsilon^2 - \sigma_\theta^2)}.$$

When $\rho = 0$, η_t is essentially transitory. It yields

$$\widehat{\phi}_{t,CM} = \frac{\rho_{\theta,\widetilde{\beta}}\sigma_{\widetilde{\beta}}}{\gamma\sigma_\theta}; \widehat{\psi}_{t,CM} = \frac{cov(\theta_i, \widetilde{\beta}_i)}{\gamma(\sigma_\varepsilon^2 + \sigma_\eta^2 - \sigma_\theta^2)}.$$

3.3 Bond economy with predictable income change

Under this market structure, agents are assumed to have only access to a one-period risk-free bond with the gross interest R which clears the capital market in the stationary equilibrium. Assume perfect annuity markets. Each agent is endowed with an initial asset a_{i0}. Only the natural borrowing constraints are imposed. Since the agent's preference satisfies the Inada condition, the natural borrowing constraints will never be binding in the equilibrium. Suppose the idiosyncratic income change is fully predictable to the agent. Since all the consumption fluctuation can be smoothed by borrowing and saving, the markets are essentially complete.

The individual i's optimization problem is:

$$\max \sum_{t=1}^{T} \varsigma_t \beta_i^{t-1} \frac{c_{i.t}^{1-\gamma}}{1-\gamma} \tag{11}$$

$$s.t. \sum_{t=1}^{T} \frac{c_{it}}{R^{t-1}/\varsigma_t} \leq \sum_{t=1}^{T} \frac{y_{it}}{R^{t-1}/\varsigma_t} + a_{i0}.$$

Take R as given, the agent i's log consumption can be thus solved as

$$\log c_{i,t} = \frac{\log(\beta_i R)}{\gamma} t - \log \sum_{t=1}^{T} [\beta_i R^{(1-\gamma)}]^{\frac{t}{\gamma}} + \log \frac{W_i}{R}, \tag{12}$$

where $W_i \equiv \sum_{t=1}^{T} \frac{y_{it}}{R^{t-1}} + a_{i0}$ is the present value of life-time wealth.

Clearly, $\log c_{it}$ is linear in age and its slope is increasing in β_i. The log consumption of agent i is increasing (decreasing) over the life cycle if $\beta_i R$ is greater (less) than 1. Consider two agents with the same life-time wealth and $\beta_i > \beta_j$, the impatient agent will consume more in the beginning, but after some time the patient agent will catch up and consume more and more afterwards. Thus consumption dispersion is expected to decrease at the very beginning, reaches zero after some time and increases since then.

3.3.1 Consumption dispersion

The variance of log consumption can be computed as

$$var(\log c_{it}) = \frac{t^2}{\gamma^2}\sigma^2_{\log\beta} + \frac{2t}{\gamma}cov(\log\beta_i, \log W_i\Omega_i^{-1}) + var(\log W_i\Omega_i^{-1}), \tag{13}$$

where $\Omega_i \equiv \sum_{t=1}^{T}[\beta_i R^{(1-\gamma)}]^{\frac{t}{\gamma}}$. Notice that the last term in the right hand side is constant over the life-cycle. The first term is increasing quadraticly with age; the second term is a linear function of age whose slope is determined by the difference between $cov(\log\beta_i, \log W_i)$ and $cov(\log\beta_i, \log\Omega_i)$. Since the life-time wealth W_i is increasing in θ_i, the higher the $cov(\beta_i, \theta_i)$, the larger the slope. Generally, we have a U-shaped consumption dispersion profile. At the beginning of life, the consumption dispersion decreases with age; if t is sufficiently large, the consumption dispersion starts to increase convexly with age.

Notice the formal resemblance of equation (13) and (9). In fact, the bond economy with fully predictable heterogeneous income profile is a special case of the complete market. The advantage of this model is that we now can express all the conditions without any knowlege of λ_i. Thus we have the following corollary:

Corollary 1 *In the bond economy with fully predictable heterogeneous income profile, the consumption dispersion is given by (13), which is a special case of that in a complete market model with stochastic income:*

(1) If $T > t^* \equiv -\gamma cov(\log\beta_i, \log W_i/\Omega_i)/\sigma^2_{\log\beta}$*, then for* $t > t^*$*, the variance of log consumption in the complete market model increases convexly with age* t*. If* $1 \le t^* < T$*, then the consumption dispersion age profile is U-shaped.*

(2) If $cov(\log\beta_i, \log W_i/\Omega_i) > -\frac{T+1}{2\gamma}\sigma^2_{\log\beta}$*, then the consumption dispersion increases over the life cycle. Therefore, the hypothesis of complete markets cannot be rejected by the observed increase of consumption dispersion over the life cycle.*

(3) Consumption dispersion over the life cycle increases with $cov(\beta_i, \theta_i)$*.*

3.3.2 Co-movement of consumption and income

In terms of BPP's estimators, the bond economy with predictable income change coincides with the complete market with stochastic income. Therefore, predictability and insurability is observationally equivalent in BPP's risk-sharing test.

Corollary 2 *In the bond economy with fully predictable heterogeneous income profile, the BPP's estimators of the pass-through of both permanent and transitory shocks are the same as those in the complete market model with stochastic income.*

Proof. *See Appendix.* ■

3.4 Matching the data

I have shown that it is theoretically possible for a complete market model to generate an increase of consumption dispersion over the life cycle and a positive co-movement of consumption and income by BPP's measure. Therefore, the significance of the previous risk sharing tests should be questioned. To go one step further, we would like to know: is it *quantitatively admissible* to match the observed increase of consumption dispersion, the co-movement of consumption and income, or even both simultaneously? While this question has to be answered in a more seriously calibrated model, here I first illustrate a tentative answer by a numerical example. It is convenient to do this in the bond economy with predictable income, a special case of complete markets, since we don't need to solve the expression for the Pareto weights.

Empirical targets

The empirical target for the consumption dispersion is from Heathcote et al. (2010a). They report the age profile of consumption dispersion for average 5 years group, starting from age 27 (average of 25 to 30) to age 57 (average of age 55 to 60). They find that the variance of log consumption rises by 0.057 over the life cycle, controlling for year effects. To directly compare with their result, I set $T = 31$.

The empirical target for the co-movement of consumption and income is naturally the pass-through of permanent shocks estimated by BPP (2008), whose estimate is 0.642 and does not have an age trend. From Propostion 2 we know that the pass-through of permanent shocks in the complete market model varies with age if $0 < \rho < 1$. But a simple calculation shows that after the first several years, its variation is almost zero. Since BPP's sample starts from age 30 and their first estimate is from age 33, we can therefore use any large t to match BPP's estimate and get the same result.

Parameters

Because this model is based on the assumption of fully predictable income process, which is different from previous empirical strategies for identifying income "shocks", it is not clear whose estimate is proper. To be consistent with time span of consumption data, I focus on the income dispersion over the life cycle in Heathcote et al. (2010a), where the variance of log earnings increases convexly by 0.30 over the life cycle. The increase of log variance of life time earnings is given by:

$$\Delta_{it} var(\log y_{it}) = \begin{cases} (T^2-1)\sigma_\theta^2 + \frac{1-\rho^T}{1-\rho}\sigma_\eta^2 & \text{for } \rho < 1 \\ (T^2-1)\sigma_\theta^2 + (T-1)\sigma_\eta^2 & \text{for } \rho = 1 \end{cases}. \tag{14}$$

I take Gueven (2009)'s estimates of $\rho = 0.821$ and $\sigma_\eta^2 = 0.029$ from his estimation of the HIP model. However, his estimate of $\sigma_\theta = 0.0195$ generates much higher income dispersion than that of Heathcote (2010a). Instead of borrowing his estimate of σ_θ, I will chose σ_θ to match the empricial income dispersion. Equation (14) implies $\sigma_\theta = 0.0169$.

I set $R = 1.04$ and $\gamma = 0.5$. I assume that θ_i and $\log\beta_i$ follow joint Normal distribution with mean $\mu_\theta = \mu_{\log\beta} = 0$.

Identification

The parameters to estimate are $\sigma_{\log\beta}$ and $\rho_{\beta\theta}$, which we have two equations for the identification:

$$\frac{T^2-1}{\gamma^2}\sigma_{\log\beta}^2 + \frac{2(T-1)}{\gamma} cov(\log\beta_i, \log W_i\Omega_i^{-1}) = \Delta_{1,T} var(\log c_{it}). \tag{15}$$

$$\frac{\rho_{\beta\theta}\sigma_{\log\beta}}{\gamma\{\sigma_\theta^2 + \rho[(\rho-1)(\rho^3-1)\sum_{\tau=1}^{t-2}\rho^{2(\tau-1)} + \rho(\rho-1)+1]\sigma_\eta^2/3\}} = \widehat{\phi}_{t,BPP}. \tag{16}$$

From equation (16), we have a restriction for the product of $\rho_{\beta\theta}$ and $\sigma_{\log\beta}$. Then, $\rho_{\beta\theta}$ and $\sigma_{\log\beta}$ can be identified using equation (15) by numerical simulation of covariance. Figure 2 plots the restrictions on $\rho_{\beta\theta}$ and $\sigma_{\log\beta}$ implied by equation (15) and (16). The circled line matches the empircal increase of consumption dispersion over the life cycle and the solid line matches the empirical co-movement of consumption and income.

[Insert Figure 2]

I also plot the stared line, where the pairs of $\rho_{\beta\theta}$ and $\sigma_{\log\beta}$ generate zero consumption dispersion over the life cycle. Those pairs which lie in the southeast region to this line are admissible for generating a life-cycle increase of consumption dispersion. This implies: given σ_β, higher $\rho_{\beta\theta}$ generates higher consumption dispersion. Interestingly, given the $\rho_{\beta\theta}$, lowering the dispersion of β will increase the dispersion of consumption! The lowest $\rho_{\beta\theta}$ for generating the observed consumption dispersion is 0.20. Therfore, the positive correlation between discount factors and income growth rates is quantitatively necessary even in generating the increase of consumption dispersion.

The solid line (equation (15)) and the circled line (equation (16)) do intersect. So, the answer to the above question of quantitative admissibility is *yes*, and *both*. The intersection of the solid line and the circled line determines a unique fair of $\sigma_{\log\beta}$ and $\rho_{\beta\theta}$. This yields $\rho_{\beta\theta} = 0.63$ and $\sigma_{\log\beta} = 0.016$. In this parsimonious complete market model, a small amount of discount rate heterogeneity and a moderate correlation between discount factors and income growth rates is sufficient to simultaneously match both the empirical increase of consumption dispersion over the life cycle and the BPP's estimate of the co-movement of consumption and income. Notice also that all the pairs on the solid line where $\rho_{\beta\theta}$ is higher than its intersection with the started line will cause the false rejection of the complete market model by the previous risk sharing tests.

3.5 Is $cov(\beta_i, \theta_i) > 0$ plausible?

For the complete market models to match the data, $cov(\beta_i, \theta_i) > 0$ is crucial. Is it empirical plausible? We cannot directly answer this question since both β and θ are unobservable, at least from any normal data set. Nevertheless, there are plenty of indirect evidence suggesting the positive correlation between β and θ.

3.5.1 Education choice

Higher education generates higher θ Empirical studies find that the group of higher education is associated with higher wage growth rate. (e.g. a recent study of PSID by Low, Meghir and Pistaferri (2010)) This is also true in cross-country: for example, Barro (1991) finds that the difference in the schooling is positively correlated with income growth rate across countries. Bils and Klenow (2000) study this effect in a human capital model and find the effect is much smaller, but still positive.

Higher education implies higher β Cagetti (2003) estimates β for different education groups using the empirical age profile of wealth and he finds that the discount factor of the college graduate is higher than that of the high school graduate, which is itself higher than that of the individuals with no high school education. Since people with higher β will have a higher wealth-income ratio in a standard incomplete market model, his result comes from the empirical fact that the group of people with higher education have a higher wealth income ratio.

If the above two observations are true, that is: people with higher discount factor tend to have higher education and people with higher education are more likely to have a higher income growth rate, then we have $cov(\beta_i, \theta_i) > 0$.

The model of education choice Higher education, of course, is endogenous. Here I will present a simple extension of the previous complete market model, which can be the micro foundation of why higher education is positively correlated with higher discount factor.

Let $V_i(\theta, \beta)$ denote the value function associated with indiviudal i's optimization problem (11) in the previous complete market model. Insert the optimal consumption rule (12) into the life-time utility function, we get that $V_i(\theta, \beta)$ is increasing in θ and β. Consider a schooling decision faced by each indiviudal at period 0, one period before she enters the labor market. θ consists of two values with $\theta_H > \theta_L$. If she chooses $E_i = 1$ and goes to school, she has to pay a schooling cost in terms of utility ω_i, and get θ_H when she starts working next period. If she chooses $E_i = 0$ and does not go to school, she will not pay any cost, and get θ_L when she starts working next period. Here we can re-interpret T as the ratio of post-schooling working years to potential non-mandatory schooling years. Agents differ in utility cost of schooling ω_i, which can be due to the difference in various factors including innate cognitive or non-cognitive ability, social environment, and finacial status, etc. The discount factor β_i and schooling cost ω_i are drawn independently from distribution F_β and F_ω, respectively.

The individual i's problem at period 0 can be solved as

$$E_i = \begin{cases} 1 & \text{if } \beta_i[V_i(\theta_H, \beta_i) - V_i(\theta_L, \beta_i)] > \omega_i \\ 0 & \text{if } \beta_i[V_i(\theta_H, \beta_i) - V_i(\theta_L, \beta_i)] \leq \omega_i \end{cases}. \tag{17}$$

There is a cut-off value of ω_i^* for each indiviudal i. She will choose *not* to go to school if $\omega_i \geq \omega_i^*$. As $\beta_i[V_i(\theta_H, \beta_i) - V_i(\theta_L, \beta_i)]$ is increasing in β_i, the cut-off value ω_i^* is decreasing in β_i. As a result, the fraction of individuals who choose schooling and thus get a higher income growth is higher in the groups of individuals with higher β, which implies $cov(\beta_i, \theta_i) > 0$. θ_i is not perfectly correlated with β_i, because the random variable of individual schooling cost ω_i, which is assumed to be not correlated with β_i, also plays a role in determining the schooling and income growth rate.

One caveat applies here. In the Mincer equation regression where we estimate the income process, education level is already controlled for. Therefore, the difference in θ should be interpreted as income growth beyond the effect from formal schooling. In my model, there is no variance of θ, if schooling is controlled for. Nevertheless, we can interpret schooling in the above model as any human capital investment which takes time, such as informal education, self-education, or any learning activity which we cannot observe in the data. Higher unobservable human capital would also increase the individual productivity and the income growth rate. Thus we could study the investment of unobservable human capital using the above model and would still get $cov(\theta, \beta) > 0$, even if education level is controlled for.

3.5.2 Mortality and health

The discount factor is, by definition, subjective. This makes the empirical identification difficult. Alternatively, the model discount factor can be interpreted in such a way that it has some objective content. Suppose the survival probability is heterogeneous in individuals. The period utility now becomes

$$u_i(c) = E\sum_{t=1}^{T}(\prod_{\tau=1}^{t}\xi_\tau)S_i^{t-1}\beta_i^{t-1}\frac{[c_i(s^t)]^{1-\gamma}}{(1-\gamma)},$$

where S_i is the idiosyncratic conditional survival rate which is constant over the life cycle and $E(S_i) = 1$. This can be interpreted as an increasing function of one's inital health level, H_i, which can be either inherited or formed in early years. Redefine $\widehat{\beta}_i \equiv S_i\beta_i$, we get the same optimization problem as (11). Therefore, any factors correlated with S_i now correlated with $\widehat{\beta}_i$, the discount factor in the model. It is natural to think that the wage growth rate is positively correlated with labor productivity, which is itself positively correlated with one's health: $\theta'(H_i) > 0$. Since $S'(H_i) > 0$, this implies $cov(\beta_i, \theta_i) > 0$. We can also use this interpretation in the previous schooling choice model, which will make the correlation between discount factor and income growth rate stronger.

4 Large Scale Models

In this section I will compare quantitatively the complete market (CM) model with the standard incomplete market (SIM) models and evaluate these models by their performances in matching the data. To this end, I will use richer and more empircally relevant models with seriously calibrated parameters. In particular, I will present a class of calibrated "large scale" models, which is an extension of the class of standard incomplete market models with overlapping-generations used in Huggett (1996), Storesletten et al. (2004), etc. By choosing the income process the agents perceive and the borrowing constraints, this model nests both complete and incomplete market structures.

4.1 The economy

Consider an economy populated by T overlapping-generations, each of which consists of a continuum of agents. Each individual enters the labor market at age 1 and can live a maximum of T periods. Agents face mortality risks. The probability of surviving between age $t-1$ and age t is denoted by ξ_t, with $\xi_1 = 1$ and $\xi_{T+1} = 0$. The measure of the new born agents is denoted by μ_1 and the population grows at a constant rate n, implying a stable population structure with $\mu_t = \mu_1 \prod_{\tau=1}^{t} \xi_\tau (1+n)^{1-t}$.

Agents enter the labor market at age 1 and the mandatory retirement age is t_R. At working age $t < t_R$, the agents supply inelastically one unit of labor, while they differ in the efficient unit of labor. The exogenous labor income of agent i is

$$y_t^i = (1-\tau) w e_t^i, \tag{18}$$

where τ is the pension tax, w is the wage rate identical to all the agents in a given cross-section, and e_t^i is the efficient unit of labor which is assumed to follow

$$\begin{aligned} \log e_{i,t} &= \kappa_t + \alpha_i + \theta_i t + z_{i,t} + \varepsilon_{i,t}, \\ z_{i,t} &= \rho z_{i,t-1} + \eta_{i,t} \ , \end{aligned} \tag{19}$$

where $\rho \in [0,1]$, $\eta_{i,t} \sim N(0, \sigma_\eta^2)$, $\varepsilon_{i,h} \sim N(0, \sigma_\varepsilon^2)$, $z_{i,1} = 0$. κ_t is the income profile which is identical to all the agents of the same age. In the stationary equilibrium where there is only a cross-section of overlapping generations, there is no time effect and the concept of cohort and age coincide. The other part is the same as the income process in the previous section. When $\eta_{i,t}$ and $\varepsilon_{i,t}$ are fully predictable and only natural borrowing constraints are imposed, it boils down to the complete market case. Therefore, it is convenient to compare the both the CM and SIM models in this one parameterized setup.

After retirement, the agent i receives pension B_{it} which is funded by a Pay-As-You-Go system through the pension tax τ. According to the U.S. Old Age pension system, the pension is a concave function of life-time average income. In computation, I will use the last period non-transitory income as a proxy of the average income to mimic the U.S. pension system.

Agents can only have access to a risk-free bond which yields the gross interest rate R. I assume that there exist perfect annuity markets for mortality risks, so that the return of asset is interest rate plus a survival premium. The agent i's budget constraint is given by

$$c_{i,t} + a_{i,t+1} \leq a_{i,t} R / \xi_t + y_{i,t}, \tag{20}$$

where $a_{i,t}$ is the asset or financial wealth. The agent can not leave negative asset at year T and faces a borrowing constraint $a_{i,t+1} \geq \underline{a}_{i,t+1}$, where $\underline{a}_{i,t+1}$ is an ad hoc borrowing constraint which can potentially a function of current state variables and can be set as low as the natural borrowing constraint. Each individual is endowed with initial wealth a_{i0}.

The interest rate is set to be exogenous, but it is straightforward to assume a production function to close the model. In that case, the interest rate is a function of the parameters of aggregate productivities.

4.2 Equilibrium

For CRRA utility function, we can obtain the balanced growth path by dividing all the quantities by the accumulated productivity growth. Given constant R and w, individual $i's$ decision problem can be written recursively as

$$V(\beta,\theta,\alpha,a,\varepsilon,z;t)=\max_{a'}\left\{c_i^{1-\gamma}/(1-\gamma)+\beta\xi_{t+1}(1+g)^{1-\gamma}E[V(\beta,\theta,\alpha,a',\varepsilon',z';t+1)|z]\right\} \tag{21}$$

subject to

$$\begin{aligned} c_i+(1+g)a' &\leq a_iR/\xi_t+\left\{\begin{array}{ll} y_t & t<t_R \\ B_t & t\geq t_R \end{array}\right. \\ a_i' &\geq \underline{a} \\ a_{T+1} &\geq 0 \end{aligned}$$

The terminal period value function is set to $V(:;T+1)=0$. The equilibrium we study is a stationary recursive competitive equilibrium where the factor prices are constant over time and the age-wealth distribution is stationary. Formally, denote $(X,B(X),\Psi_t)$ as the probability space, where X is the domain of state variables, $B(X)$ is the Borel σ-algebra on X, and Ψ_t is the probability measure. Denote $P(x,t,B)$ as the probability that an age t agent transit to set B given the agent's current state is x. This transition function is derived from the individual's decision rule $a'(\cdot)$.

Definition 1 *A stationary recursive competitive equilibrium with exogenous R and w is a pair of value function and a decision rule $\{V(\cdot),a'(\cdot)\}$, such that*

(i) Given R and w, $V(\cdot),a'(\cdot)$ solve the agent's Bellman equation (21).

(ii)The distribution is consistent with individual's behavior: $\Psi_{t+1}=\int_X P(x,t,B)d\Psi_t$, for all t and $B\in B(X)$.

(iii)Pension is funded by a Pay-As-You-Go system: $\tau\sum_{t=1}^{t_R-1}\mu_t\int_X y_t^i d\Psi_t=\sum_{t=t_R}^{T}\mu_t\int_X B_t^i d\Psi_t$

.

Under this standard setup, the stationary equilibrium is attained immediately. The individual's problem can be solved numerically by backward induction using Euler equation. To speed up the algorithm, I use the endogenous grid method developed by Carrol (2006). I use linear interpolation with 71 grid points for the positive asset values and 50 grids for the negative asset values. Grids on positive asset values are formed triple exponentially to make more grids where asset level is lower. I use two discrete states for each of the exogenous state variables $(\gamma, \theta, \varepsilon, \eta)$. 50,000 agents are used in the simulation. In the unit root income process, I use 41 state spaces for the permanent component. In the AR(1) income process, I discretize the persistent shock by a 41 state Markov chain using the method suggested by Tauchen (1986).

4.3 Calibration

4.3.1 Externally calibrated parameters

Demography The model period is 1 year. Agents begin to work at age 22, which coincides with age 1 in the model. Conditional on surviving, they then work for 45 years, retire at age 66 and die at age 100. Agents or individuals are interpreted as households in the data, and hence we chose the conditional surviving rate from the U.S. life table for *females* in 1989-1991. The annual population growth rate is set to $n = 1.0\%$ per year.

Preference The risk aversion is set to $\gamma = 2$. For the estimation of the distribution of β, I will discuss later on.

Income process The secular productivity growth rate is set to $g = 1.5\%$ per year. R is set to 1.04. The average age profile of income κ_t is chosen to match the average income in the U.S. Census 1990.

In the SIM models, I use both RIP and HIP. In the RIP model, I use Storesletten et al. (2004)'s estimation. The coefficient of auto-regression ρ is very close to 1, which is 0.98. And I will set it to 1 in computation and therefore the persistent shock is actually permanent. The variance of the fixed effect, persistent shock, transitory shocks is $\sigma_\alpha^2 = 0.2105$, $\sigma_\eta^2 = 0.0161$, $\sigma_\varepsilon^2 = 0.0630$, respectively. In the HIP model, I use the income process estimated by Guvenen (2007, 2009). The variance of the wage growth rate is $\sigma_\theta^2 = 0.00038$. The correlation coefficient between the idiosyncratic part of wage growth and the fixed effect is $cov(\theta_i, \upsilon_i) = -0.002$. The variance of the fixed effect, persistent shock and transitory shock is $\sigma_\upsilon^2 = 0.022$, $\sigma_\eta^2 = 0.029$, $\sigma_\varepsilon^2 = 0.047$, respectively. Unlike Guvenen (2007)'s learning story, I assume there is no prior uncertainty about θ_i and thus the agent has complete information of her income growth rate when she enters the labor market.

In the CM model, I will use the HIP process by Guvenen (2007)'s estimation, but assume all the individual income changes are predictable to the agent.

Pension The pension system in the benchmark model is designed to mimic the U.S. Old

Age pension system as follows:

$$B_i = \lambda \times \begin{cases} 0.9 y_{i,p,R-1} & \text{for } y_{i,p,R-1} < 0.3\overline{y}_{p,R-1} \\ 0.27\overline{y}_{p,R-1} + 0.32(y_{i,p,R-1} - 0.3\overline{y}_{p,R-1}) & \text{for } \overline{y}_{i,p,R-1} \in (0.3\overline{y}_{p,R-1}, 2\overline{y}_{p,R-1}] \\ 0.81\overline{y}_{p,R-1} + 0.15(y_{i,p,R-1} - 2\overline{y}_{p,R-1}) & \text{for } \overline{y}_{i,p,R-1} \in (2\overline{y}_{p,R-1}, 4.1\overline{y}_{p,R-1}] \\ 1.1\overline{y}_{p,R-1} & \text{for } \overline{y}_{i,p,R-1} > 4.1\overline{y}_{p,R-1} \end{cases}$$

Different from Storesletten et al. (2004), I use the last working year income excluding the transitory part instead of life-time average income. Guvenen (2007) uses a similar expression where the last period income serves as the proxy for the average life-time income. I exclude the transitory part because the last period transitory part gives us little information of the average life-cycle income and the non-transitory part is still highly correlated with average life-cycle income. I re-scale the pension system to make the replacement ratio of the model match that of the U.S. data, which is 0.48. It generates 0.92 in the benchmark RIP model and 0.82 in the HIP model. The pension tax τ can be solved directly by the PAYG system, which is 0.1325. I will also discuss the extreme case when there is no pension.

Borrowing constraint In the benchmark SIM models, the households are allowed to borrow up to the expected income of next year $\underline{a}_{i,t+1} = -E_t(y_{i,t+1})$, which is the same as used in Storesletten et al. (2004). I also consider two other extreme cases: one is that all households are excluded from any borrowing, i.e. $\underline{a} = 0$; the other is that no ad hoc borrowing constraints is imposed and I only impose the terminal condition that agents cannot die in debt at age $T+1$. In other words, I set the borrowing constraint as low as the natural borrowing constraint which is not binding in household's optimal solution with CRRA preference.

In the CM, only natural borrowing constraints are imposed.

Initial wealth The initial wealth distribution is calibrated to mimic the wealth distribution of households under age 25 in SCF 1992 (Diaz-Gimenez et al. 1997). I approximate the initial wealth distribution by a log normal distribution whose mean is set to match the initial wealth/income ratio, which is 0.89, and then I calibrate its variance to match the wealth Gini for those young households, which is 0.87.

The above parameters for calibration are summarized in Table 1.

[Insert Table 1]

4.3.2 Distribution of discount factors

In a standard model without discount rate heterogeneity, the variance of β_i is restricted to be zero and the conventional procedure of calibrating β is to minimize the distance between the wealth/income ratio in the simulated model and that of the data. In the model with discount rate heterogeneity, however, we need more moments to identify the distribution of β_i. In this paper, I will follow Hendricks (2007a) to use wealth Gini coefficient as the additional moment[4].

I approximate the distribution of discount factors F_β by a discrete distribution with two values $\beta_l = \overline{\beta}(1 - \Delta\beta)$ and $\beta_h = \overline{\beta}(1 + \Delta\beta)$. The probability of being a relatively impatient agent is p_l, and the probability of being a relatively patient household is $p_h = 1 - p_l$. To reduce the parameters for calibrating, the middle value of the discount rate, $\overline{\beta}$, is assumed to be equal to the calibrated value of β in a standard model without discount rate heterogeneity to match the wealth income ratio in the U.S. of 3.1 from SCF 1992 (Diaz-Gimenez et al. 1997). Alternatively, I will also consider the wealth/income ratio of 4.56, which is the average of SCF 1992 and 1998 and is used in Sun (2010). The two targets for estimating $\Delta\beta$ and p_l are the wealth income ratio again and the Gini coefficient of the households at retirement age. To save the time for computation, $\Delta\beta$ is assumed to lie on 30 grids, from 0.005 to 0.15, and p_l is assumed to lie on 100 grids, from 1% to 99%. $\Delta\beta$ and p_l are chosen to minimize the loss function which is the sum of the absolute value of the percentage deviation between data and model in these two targets.

In the complete market setup, both σ_β and $cov(\beta_i, \theta_i)$ would affect the wealth inequality. At the moment, we restrict $cov(\beta_i, \theta_i)$ to be zero. If $cov(\beta_i, \theta_i) \geq 0$, this estimate of σ_β should be interpreted as the higher bound of σ_β.

The estimations of β_i are summarized in Table 2. The mean and standard deviation of β_i is not far from, $\overline{\beta}$ and $\Delta\beta$.

[Insert Table 2]

The result shows that in the benchmark RIP and HIP models, $\Delta\beta$ is 0.030 and 0.035, p_l is 64% and 66%, respectively. In general, the estimates for RIP models are close to that for the HIP models. The middle value of discount factors, $\overline{\beta}$, is higher in HIP model than in RIP model, because less persistent shock reduces the precautionary motive for saving and thus lowers the wealth income ratio. To match the wealth income ratio, $\overline{\beta}$ has to be increased.

Another approach is to directly estimate β_i using Euler equation (Alan and Browning 2010). However, this estimation attributes all inter-temporal wedges into discount factors. For comparison, I will also consider the estimates of mean and variance of β_i from the estimation

[4] In Cagetti and De Nardi (2005), they show that an incomplete market model with entrepreneur behavior and bequest motive can account for the empirical wealth distribution. Notice that different discount factors imply different saving rates for different groups of agents, which is consistent with the very high saving rate of the entrepreneurs in their model.

of Hendricks (2007a) and Alan and Browning (2010)[5]. The estimation of distribution of β_i in both RIP and HIP model are very close to the estimation by Hendricks (2007a).

The estimation result of $\Delta\beta$ and p_l in the complete market model shows that it requires much less dispersion of β_i to generate the empirical observed wealth distribution. Only $\Delta\beta$ = 1.7% and $p_l = 55\%$ is needed. It is because in the complete market people hold less wealth than in the incomplete market, since there is no precautionary saving motive. When the agents are old, they would not accumulate a large amount of life time wealth due to full risk sharing (as there is no risk at all).

5 Quantitative Results

5.1 Consumption dispersion

5.1.1 SIM models

Figure 3 shows the age profiles of consumption dispersion in both RIP and HIP benchmark models from age 25 to age 65. Since we are interested in the increase of consumption dispersion, not its levels, I normalize the consumption dispersion for age 25 to be zero. The main results for SIM models are as follows:

First, the magnitude of the increase of consumption dispersion is very high in both RIP and HIP models, compared with Heathcote (2010a). Although all the SIM models generate a significant increase of consumption dispersion over the life cycle, these models become less plausible because they generate too much consumption dispersion.

Second, with $\sigma_\beta > 0$, in the RIP model there is almost no difference in the increase of consumption dispersion over the life cycle. In the HIP model, $\sigma_\beta > 0$ does increase the consumption dispersion, but the magnitude is very small.

Third, in the data, the consumption dispersion increases very slowly, even U-shaped, in the begining of the life cycle. In the SIM models, the consumption dispersion increases rapidly at the very beginning. With discount rate heterogeneity, the life cycle profiles of consumption dispersion are tilted up, more so from age 50 on. In the RIP model, it makes the profile more convex in the last ten working years. In the HIP model with discount rate heterogeneity, the age profile of consumption dispersion becomes approximately linear.

[5] I take the estimation of β in the no bequests model of Hendricks(2007a Table 2). Since he use 5 grids for β_i and set β to be 2% or 6% lower or higher than $\overline{\beta}$, I set p_l to the fraction of the agents with 0.94 or 0.98 $\overline{\beta}$ plus half of the fraction of agents with $\overline{\beta}$, which gives $p_l = 61\%$. I set σ to be half of the difference between weighted average of the lowe βs and the high βs, which is 0.035. For Alan and Browning (2010), I assume a symmetric distribution and take its coefficient of variance as σ, which is 0.09. In both cases, I recalibrate the $\overline{\beta}$ to match the wealth income ratio.

5.1.2 CM model

Figure 4 plots the age profile of consumption dispersion in the CM models with $cov(\beta_i, \theta_i) = 0$ and $cov(\beta_i, \theta_i) > 0$. They are U-shaped as we expected. If $cov(\beta_i, \theta_i) = 0$, the variance of log consumption starts to increase only after the middle age and does not generate an overall increase over the life cycle. This result echoes the previous results in the SIM models with $\sigma_\beta > 0$, where its effect on the magnitude of increasing consumption dispersion is negligible. If $cov(\beta_i, \theta_i) > 0$, the variance of log consumption will increase more. As the first step, I will not estimate $cov(\beta_i, \theta_i)$, but instead ask: how much $\rho_{\beta,\theta}$ do we need to generate an empirical observed increase of consumption dispersion? In the benchmark CM model, to match the data it requires $\rho_{\beta,\theta} = 0.20$.

In short, with the presence of $cov(\beta_i, \theta_i) > 0$, the calibrated complete market model can generate an empirical plausible increase of consumption dispersion. Therefore, the risk sharing tests by consumption dispersion doe not give us enough information to identify either the market completeness or the nature of income shocks.

5.1.3 Understanding the role of $\sigma_\beta > 0$ in SIM

In the standard incomplete market models, the result of the small impact of $\sigma_\beta > 0$ on the increase of consumption dispersion needs more explorations. Is it only due to the choice of calibrated parameters or is there something more fundamental? To single out the effect of $\sigma_\beta > 0$, I shut down all the income shocks and contribute all the increase of variance of log consumption to the heterogeneity discount factors. This is very much like the complete market model, with the only difference in borrowing constraints. Figure 5 and 6 show the decomposition of consumption dispersion in both RIP and HIP models. The age profile of consumption dispersion is U-shaped, but the magnitude is small relative to the large increase of consumption dispersion with income shocks.

Let us decompose the consumption dispersion into two parts: within groups of the same discount factor and between groups of different discount factors.

$$Var(\log c) = \underbrace{Var(E(\log(c))}_{\text{Between Group}} + \underbrace{E(Var(\log c))}_{\text{Within Group}} \tag{22}$$

Figure 7 and 8 plot the consumption dispersion within each household group. The $var(\log c)$ of the patient agents increase as usual, but $var(\log c)$ of the impatient agents will decrease when retirement is approaching. This is an unexpected result. Why? Let us move to Figure 9 and 10, where the mean of log consumption is shown. In this economy the low β households are very impatient, the same as the agents in Gourinchas and Parker (2002)'s economy. Their consumption profile is thus hump-shaped and they will consume less and less to accumulate retirement wealth when old. The variance of log consumption decreases mechanically because of the sharp decrease of the mean log consumption. When young, the first term is decreasing (recall the U- shape result), the second term is increasing. When old, the first term is increasing but the second term is decreasing. Therefore, the effect discount rate heterogeneity is mitigated.

It is also helpful to understand the role of $\sigma_\beta > 0$ and its interactions with other important ingredients of the model, such as pension, borrowing constraints and wealth/income ratio. The discussion is in the Appendix.

5.2 Co-movement of consumption and income

5.2.1 SIM models

Table 4 shows the consumption insurance by BPP's measure. The main findings are as follows:

[Insert Table 4]

First, the SIM models with RIP generate too little insurance while the SIM models with HIP generate two much insurance, in terms of the pass-through of permanent shocks.

Second, the presence of discount rate heterogeneity lowers the consumption insurance by BPP's measure and thus the empirical finding of the degree of risk sharing is down-ward biased. In other words, there could exist higher consumption insurance than that found by BPP, if there is discount rate heterogeneity. It brings HIP model closer to the data, because HIP with less persisient shocks is in fact more insurable. As to the transitory shocks, with $\sigma_\beta > 0$ I find that ψ is much higher than the data and not significantly from zero in both HIP and RIP models. This is because the low β households on average have lower wealth and therefore have less ability of self-insurance even against the transitory shocks.

Third, as plotted in Figure 11, all the SIM models have a downward age trend of permanent pass-throughs. This is counter-factual, as BPP find no age-trend. Adding discount rate heterogeneity does not change the slope very much.

In short, although the SIM model with RIP and $\sigma_\beta = 0$ matches the transitory pass-through, all the SIM models are rejected by both the level of permanent pass-throughs and the shape of its life cycle profile.

5.2.2 CM Model

Matching BPP In CM model we can either solve $\widehat{\phi}_{t,CM}$ and $\widehat{\psi}_{t,CM}$ from the analytical expression or from the result of the simulated model, both of which should give us the same result.

The BPP estimates of benchmark CM model are reported in Table 4. The $\widehat{\phi}_{t,CM}$ in the benchmarke CM model is 0.06, which is positive but far from BBP's estimation result of 0.642. As to the transitory shocks, they are nearly zero. If there is measurement error, then it is not significantly different from BPP's estimate of 0.05. The reason why the benchmark calibration cannot fully account for the consumption insurance is as follows:

The covariance between β_i and θ_i consists of three elements: $\rho_{\beta,\theta}, \sigma_\beta$ and σ_θ. Remember in the benchmark calibration that we chose σ_β to match the empirical wealth distribution and $\rho_{\beta,\theta}$ to match the increase of consumption dispersion over the life cycle. Since both σ_β and $\rho_{\beta,\theta}$ are relatively small in my estimation, it is hard to generate a large $\widehat{\phi}_{t,CM}$. To see this clearly, let us analyze two polar cases with $\rho = 1$ or $\rho = 0$. Combine this two expressions and rewrite it as:

$$\rho_{\beta,\theta}\sigma_\beta \approx \gamma\widehat{\phi}_{t,CM}(\sigma_\theta + \frac{\rho\sigma_\eta^2}{3\sigma_\theta});\ \text{for } \rho = 1 \text{ or } 0. \tag{23}$$

To match the data we need $\rho_{\beta,\theta}\sigma_\beta = 1.284(\sigma_\theta + \frac{\rho\sigma_\eta^2}{3\sigma_\theta}) > 1.284\sigma_\theta$, which means σ_β must be higher than $\sigma_{\theta.}$ As I use $\sigma_\theta = 0.0195$, it is not surprising that the CM model cannot match BPP with $\sigma_\beta = 0.017 < \sigma_\theta$.

The nice property of the analytical expression of $\widehat{\phi}_{t,CM}$ is that we do not need to solve the simulated model to get the result. Since $\rho_{\beta,\theta}, \sigma_\beta$ or even ρ and σ_θ are not observable to econometricians, the estimate from $\widehat{\phi}_{t,CM}$ would give us indirect inference for $\rho_{\beta,\theta}, \sigma_\beta$ or even ρ and σ_θ. In the benckmark CM model, we have $\rho_{\beta,\theta}\sigma_\beta = 0.0034$. To match the empirical pass-through of permanent shocks 0.642, we directly get $\rho_{\beta,\theta}\sigma_\beta = 0.036$.

The life cycle profile of $\widehat{\phi}_{t,CM}$ in the model is in Figure 12. After the first few years, the model $\widehat{\phi}_{t,CM}$ is almost constant. Since BPP's estimates starts from age 33, this is a feature consistent with the empirical finding, which the SIM models fail to match.

It should be noticed that $\rho_{\beta,\theta}$ and σ_β cannot be identified using the co-movement of consumption and income, as we have already seen in the numerical example in Section 3. The estimation strategy would be: given the restrictions of $\rho_{\beta,\theta}\sigma_\beta$, choosing the joint distribution of $F(\beta, \theta)$ to match wealth distribution (first and second moments) in the previous analysis. Here I consider two polar cases: one is to set $\rho_{\beta\theta} = 1$ and get the lower bound of $\sigma_\beta = 0.036$, which is close to the benchmark estimation of HIP model; the other is the estimate of $\sigma_\beta = 0.09$ from Alan and Browning (2003), which is the upper bound of all my estimates;this yields $\rho_{\beta,\theta} = 0.40$. In the calibrated CM model, both estimates generate less wealth Gini and larger increase of log variance than the data. Nevertheless, the $\rho_{\beta,\theta}\sigma_\beta$ can be lowered if we have difference choices of two importance factors: γ and ρ. Notice in the benchmark calibration we chose $\gamma = 2$ and $\rho_{\beta,\theta}\sigma_\beta$ is very sensitive to the elasticity of substitution. The ρ I use is Guvenen (2009)'s estiamte of 0.821. If the econometric income process model is mis-specified, as it is in the complete market model, the choice of ρ should be questionable. When ρ is lower, θ_i would have a much larger impact on the individual's income and thus generate a higher co-movement of consumption and income.

It is not surprising that the extreme case of complete market model, when well calibratred, fails to exactly match the data. Although it can match successfully either the observed consumption dispersion or the co-movement of consumption and income, it would generate too much increase of consumption dispersion if it is to account for all the observed co-movement of consumption and income. Nevertheless, this calibrated exercise gives us important messages about the role of ex ante heterogeneity: the covariance between income growth rates and discount factors plays a more important role than the dispersion of discount factors alone.

Finally, we should notice that BPP's measure is sensitive to model mis-specifications: the identification of the pass-through parameters hinges on the model specification of the income process. A model-free method would be a robust check, as BPP reported the covariance of income and consumption. However, when $cov(\beta_i, \theta_i) > 0$, this "robust" procedure will also cause false rejection of the complete market model, since all the co-movement of consumption and income may come from the covariance of discount factors and income growth rates.

Evaluating the models The quantitative results show that the SIM models and the CM model perform very differently. The key difference of their performance in matching the data can be summarzied as follows:

1. The SIM models generate too much consumption dispersions over the life cycle and there is no way to lower it to its empirical counterpart; the CM model can account for the observed consumption dispersion over the life cycle.

2. The SIM models generates a monotonic increase of consumption dispersion over the life cycle. In the data, it is U-shaped for the first 10 years. The CM model generate a U-shaped age profile of consumption dispersion.

3. The SIM models with RIP generate too much co-movement of consumption and income, while the SIM models with HIP generate too little. The CM model with positive covariance of discount factors and income growth rate can account for the observed co-movement of consumption and income.

4. The SIM models generate a down-ward sloping age trend of BPP's permanent pass-throughs. There is no such trend in the data. The CM model generates an almost flat age

profile of BPP's permanent pass-throughs.

In all the above dimensions, the CM model outperforms the SIM models. Although the calibrated CM model cannot account for all the data targets simultaneously, neither can the SIM models. It is therefore premature to reject CM model too quickly by the previous risk sharing tests. Due to its simplicity, the CM model provides us with a useful framework for risk sharing: by understanding the success and failure of the calibrated CM model in matching the data, we can learn more about the market structure, the income process and the preference heterogeneity.

6 Conclusion

This paper revisits the risk sharing tests which use micro data sets of consumption and income distributions. These tests reject the hypothesis of complete markets because they find that consumption dispersion increases over the life cycle and individual consumption co-moves with income, which cannot be reconciled with any standard complete market (CM) models.

I extend the standard CM model to allow positive correlation between heterogeneous discount factors and heterogeneous income growth rate, a feature consistent with the implications of an education choice model or a health model where the discount factor incorporates the survival rate. The main results are: it is not only theoretically possible, but also quantitatively admissible for a simple complete market model to account for the empirical evidence on both the consumption dispersion and the co-movement of consumption and income. In a calibrated framework, the CM model outperforms the standard incomplete market (SIM) models in the sense that the results from the CM model are closer to the data in both the magnitude and the shape of the life cycle profiles. Therefore, I conclude that the emprical findings from the previous risk sharing tests using micro data sets from consumption and income distributions are not sufficient for testing market completeness.

In the "heterogeneous agent" literature in macroeconomics, agents are ex ante homogeneous in most aspects. This paper shows that the incomplete market model with only ex post heterogeneity and the complete market model with ex ante heterogeneity can be observationally equivalent in the previous risk sharing tests. This result is the first step towards future research in two directions: one is to design a more proper risk sharing test which does not merely rely on the information of either the consumption dispersion or the co-movement of consumption and income. The other is to estimate the ex ante heterogeneity to account for the empirical plausible consumption and income distributions, starting from a simple complete market framework, which could give us a better understanding of the relative importance of ex ante heterogeneity.

In this sense, this paper is related to the literature of evaluating ex ante and ex post source of lifetime inequality as in Keane and Wolpin (1997) and Huggett et al. (2010).

When there is discount rate heterogeneity, the estimated degree of of consumption insurance is down-ward biased. Even if we do not take the extreme case of complete market, it suggests that there could exist more risk sharing, or more insurable risks, than we previously thought. In this sense, this paper is also related to models where there is more risk sharing than the SIM models and less risk sharing than the CM models, e.g., Kruger and Perri (2006), Attanasio and Pavoni (2009), Heathcote, Storesletten and Violante (2009), Ales and Maziero (2009), etc. For example, as similar discount factors could imply similar other individual characteristics, this paper presents a new mechanism of why the between-group consumption dispersion follows more closely to income inequality than the with-in group consumption dispersion does, which is complementary to Kruger and Perri (2006)'s limited enforcement explanation.

Using micro data to calibrate the preference parameters in quantitative models, as Browning et al. (1999) point out, may have potential flaws and has to be taken carefully, especially when there is preference heterogeneity. This paper, among a few others, estimates the distribution of discount factors from the simulated model. It would be interesting to calibrate the joint-distribution of discount factors and income growth rates from micro data if we can find new ways of identification. Due to its simplicity, the CM model provides us with a useful framework for risk sharing: by understanding the success and failure of the calibrated CM model in matching the data, we can learn more about the market structure, the income process and the preference heterogeneity. It would be interesting to estimate the discount rate heterogeneity, the income growth rate heterogeneity and their correlation simultaneously in one quantitative framework, using the data from income growth, consumption growth, education and health status. I leave it to future research.

7 Appendix

7.1 Proofs

Proof. (Proposition 1)

The first result comes straightforwardly from the property of quadratic function. The second result comes from checking $\Delta_{1,T} var \log c_{i,t} > 0$. ■

Proof. (Proposition 2)

The "true" pass-through of permanent shocks to consumption growth can be done by OLS: Since $cov(\Delta\widehat{c}_t, \eta_t) = cov(\Delta\widehat{c}_t, \varepsilon_t) = 0$ in complete market, $\phi = \psi = 0$. From the data, the econometricians can not observe η_t and ε_t. BPP suggest instrument estimation to get the variance and covariance needed above. The variance of permanent shocks is identified as:

$$\begin{aligned}
& E[\Delta\widehat{y}_t(\Delta\widehat{y}_{t-1}+\Delta\widehat{y}_t+\Delta\widehat{y}_{t+1})] \\
= \; & E[(\theta_i+\rho(\rho-1)z_{t-2}+(\rho-1)\eta_{t-1}+\eta_t+\varepsilon_t-\varepsilon_{t-1}) \\
& (3\theta_i+\eta_{t+1}+\rho\eta_t+\rho^2\eta_{t-1}+(\rho^3-1)z_{t-2}+\varepsilon_{t+1}-\varepsilon_{t-2}) \\
= \; & 3\sigma_\theta^2+\rho(\rho-1)(\rho^3-1)var(z_{t-2})+[\rho^2(\rho-1)+\rho]\sigma_\eta^2 \\
= \; & 3\sigma_\theta^2+\rho[(\rho-1)(\rho^3-1)\sum_{\tau=1}^{t-2}\rho^{2(\tau-1)}+\rho(\rho-1)+1]\sigma_\eta^2
\end{aligned}$$

The covariance of consumption growth and permanent shocks is identified as

$$\begin{aligned}
& E(\Delta\widehat{c}_t(\Delta\widehat{y}_{t-1}+\Delta\widehat{y}_t+\Delta\widehat{y}_{t+1})) \\
= \; & E(\frac{\widetilde{\beta}_i}{\gamma}+\Delta_t\Gamma_t)(3\theta_i+\eta_{t+1}+\rho\eta_t+\rho^2\eta_{t-1}+(\rho^3-1)z_{t-2}+\varepsilon_{t+1}-\varepsilon_{t-2}) \\
= \; & \frac{3}{\gamma}cov(\log\beta_i,\theta_i)
\end{aligned}$$

The last step uses the fact that $\Delta_t\Gamma_t$ is not individual i specific. Therefore,

$$\begin{aligned}
\widehat{\phi}_{t,CM} &= \frac{E(\Delta\widehat{c}_t(\Delta\widehat{y}_{t-1}+\Delta\widehat{y}_t+\Delta\widehat{y}_{t+1}))}{E(\Delta\widehat{y}_t(\Delta\widehat{y}_{t-1}+\Delta\widehat{y}_t+\Delta\widehat{y}_{t+1}))} \\
&= \frac{cov(\log\beta_i,\theta_i)}{\gamma\{\sigma_\theta^2+\rho[(\rho-1)(\rho^3-1)\sum_{\tau=1}^{t-2}\rho^{2(\tau-1)}+\rho(\rho-1)+1]\sigma_\eta^2/3\}}
\end{aligned}$$

The BPP's test of complete market comes from the fact that if $\sigma_\beta=0$, we get $cov(\log\beta_i,\theta_i)=0$ and thus $\widehat{\phi}_{t,CM}=0$. But the empirical finding of $\widehat{\phi}_{t,CM}>0$ can not reject the hypothesis of complete markets, if $cov(\log\beta_i,\theta_i)>0$.

The variance of transitory shocks is identified as:

$$\begin{aligned}
& E(\Delta\widehat{y}_t\Delta\widehat{y}_{t+1}) \\
= \; & E[(\theta_i+(\rho-1)z_{t-1}+\eta_t+\varepsilon_t-\varepsilon_{t-1}) \\
& (\theta_i+\rho(\rho-1)z_{t-1}+(\rho-1)\eta_t+\eta_{t+1}+\varepsilon_{t+1}-\varepsilon_t)] \\
= \; & \sigma_\theta^2-\sigma_\varepsilon^2+\rho(\rho-1)^2var(z_{t-1})+(\rho-1)\sigma_\eta^2 \\
= \; & \sigma_\theta^2-\sigma_\varepsilon^2+(\rho-1)[\rho(\rho-1)\sum_{\tau=1}^{t-1}\rho^{2\tau}+1]\sigma_\eta^2
\end{aligned}$$

The covariance of consumption growth and transitory shocks is identified as

$$\begin{aligned}
& E(\Delta\widehat{c}_t\Delta\widehat{y}_{t+1}) \\
= \; & E[(\frac{\log\beta_i}{\gamma}-\Delta_t\Gamma_t)(\theta_i+(1-\rho)z_t+\eta_{t+1}+\varepsilon_{t+1}-\varepsilon_t)] \\
= \; & \frac{1}{\gamma}cov(\log\beta_i,\theta_i)
\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{\psi}_{t,CM} &= -\frac{E(\Delta\widehat{c}_t\Delta\widehat{y}_{t+1})}{E(\Delta\widehat{y}_t\Delta\widehat{y}_{t+1})} \\ &= \frac{cov(\log\beta_i,\theta_i)}{\gamma\{\sigma_\varepsilon^2-\sigma_\theta^2-(\rho-1)[\rho(\rho-1)\sum_{\tau=1}^{t-1}\rho^{2\tau}+1]\sigma_\eta^2\}}\end{aligned}$$

■

Proof. (Corollary 2)

In the incomplete market with predictable income change, the true pass-throughs are not well-defined because there are no shocks at all. The BPP instrument estimation of ϕ can still be calculated using $E[\Delta\widehat{c}_t(\Delta\widehat{y}_{t-1}+\Delta\widehat{y}_t+\Delta\widehat{y}_{t+1})]$, $E(\Delta\widetilde{c}_t\Delta\widehat{y}_{t+1})$, $E[\Delta\widehat{y}_{t+1}(\Delta\widehat{y}_{t-1}+\Delta\widehat{y}_t+\Delta\widehat{y}_{t+1})]$ and $E(\Delta\widehat{y}_t\Delta\widehat{y}_{t+1})$. The last two terms are only functions for income process, so they are the same in both models. The consumption growth rate in this model is derived as

$$\Delta\widehat{c}_t = \frac{\log\beta_i+\log R}{\gamma}. \tag{24}$$

Recall in the complete markets case $\Delta\widehat{c}_t = \frac{\log\beta_i}{\gamma} - \Delta_t\Gamma_t$. The difference of $\Delta\widehat{c}_t$ in two models is only an additional aggregate term which is orthogonal to any individual income change. Therefore, $E[\Delta\widehat{c}_t(\Delta\widehat{y}_{t-1}+\Delta\widehat{y}_t+\Delta\widehat{y}_{t+1})]$ and $E(\Delta\widehat{c}_t\Delta\widehat{y}_{t+1})$ are the same in both models, which yield the same BPP's estimators. ■

7.2 Robustness Check

7.2.1 $\sigma_\beta > 0$ and pension

Figure 13 and 14 show the age profile of consumption dispersion without pension. The shape of consumption dispersion becomes more convex when pension is excluded. In other words, including pension mitigates the effect of discount rate heterogeneity, especially for the old agents.

Excluding pension will make old household more willing to save for retirement. To match the wealth distribution, it requires higher heterogeneity of discount rate and higher fraction of relatively impatient households. As we have seen in Table 2, it gives $\sigma = 0.085$, 0.080 and $p_l = 81\%$, 78% in RIP and HIP models, respectively. This drives up the consumption dispersion between different household groups.

7.2.2 $\sigma_\beta > 0$ and borrowing constraint

As to the sensitivity of borrowing constraints, I consider two extreme cases: no borrowing and natural borrowing constraints (NBC). In Figure 13 and Figure 14, we can see the effect of excluding borrowing is to tilt up the consumption dispersion profile and is very close to the profile of no-pension economy for most part of the life cycle. This effect is the same as the effect of pension, which makes the households save more and thus requires higher degree of heterogeneity discount rate to match the data. However, the shape of profile in the no borrowing case differs from the profile without pension in that the former becomes concave in the last 10 working years.

Why tight borrowing constraints may cause the heterogeneous discount rate model give the opposite implication for consumption dispersion for the old households?

The low β households the consumption dispersion decreases when approaching retirement. They are willing to borrow, although they are not allowed to do so, which causes the borrowing constraints of those agents to be binding. Consumption for the borrowing constrained households is lower than that derived from the optimal consumption rule when the borrowing constraint is not binding. When approaching retirement, some of the previously borrowing constrained agents become unconstrained. Since those are the households with relatively lower consumption level, the with-in group consumption dispersion decreases.

When there are only natural borrowing constraints, in the RIP model the result does not differ much from the benchmark case, because the borrowing constraint of expected next year's income is hardly binding for any households. In the HIP model, some agents with lower slope of wage are more likely to hold zero or negative wealth and the borrowing constraints in the benchmark model are more often binding. Hence, the consumption dispersion profile goes in the opposite direction as the no borrowing case. It is U-shaped, which is similar to the result in the complete market with no idiosyncratic shocks.

There is no consensus for the specification of borrowing constraint. It would not be a problem for the standard model without discount rate heterogeneity, since the fact that in the model no one will be borrowing constrained near retirement makes the borrowing constraints quantitatively less important for consumption dispersion. If there exists discount rate heterogeneity, however, the assumption that there is no borrowing is not innocuous and therefore the borrowing constraint should be chosen more carefully.

7.2.3 $\sigma_\beta > 0$ and wealth-income ratio

In Figure 13 and Figure 14, although the choice of wealth-income ratio changes the distribution of β, it gives negligible effect on the consumption dispersion profile. This result is different from the standard model. It is because higher wealth-income ratio has two effects. Normally it increases $\overline{\beta}$ and thus makes the household holding more wealth for self-insurance, which lowers the consumption dispersion. On the other hand, when $\overline{\beta}$ is high, $\Delta\beta$ has to be increased to match wealth distribution, which increases the consumption dispersion. The results show that these two effects almost cancel each other out, and therefore the consumption dispersion is not very sensitive to the choice of wealth-income ratio.

References

[1] Alan, Sule and Martin Browning (2010), "Estimating Inter-temporal Allocation Parameters using Simulated Residual Estimation," *Review of Economic Studies* **77(4)**, 1231-1261.

[2] Ales, Laurence and Pricila Maziero (2009), "Accounting for Private Information," Mimeo.

[3] Attanasio, Orazio, Erich Battistin and Hidehiko Ichimura (2007), "What Really Happened to consumption dispersion in the United States?," Mimeo.

[4] Attanasio, O., E. Battistin, and M. Padula (2010), "The Evolution of Consumption and Wage Inequality: Evidence and Implications for Consumption Smoothing," Mimeo.

[5] Attanasio, Orazio and Steven Davis (1996), "Relative Wage Movements and the Distribution of Consumption," *Journal of Political Economy* **104**(6), 1227-62.

[6] Attanasio, Orazio and Nicola Pavoni (2009), "Risk Sharing in Private Information Models, with Asset Accumulation: Explaining the Excess Smoothness of Consumption," Mimeo.

[7] Altonji, Joseph, Fumio Hayashi and Laurence Kotlikoff (1992), "Is the Extended Family Altruistically Linked? Direct Tests Using Micro Data," *American Economic Review* **82**(4), 1177-98.

[8] Badel, Alejandro and Mark Huggett (2010), "Interpreting Life-Cycle Inequality Patterns asan Efficient Allocation: Mission Impossible?" Mimeo.

[9] Barro, Robert J. (1991) "Economic Growth in a Cross Section of Countries," *Quarterly Journal of Economics*, **106**(2), 407-44.

[10] Barsky, Robert, Thomas Juster, Miles Kimball and Matthew Shapiro, "Preference Parameters and Behavioral Heterogeneity: An Experimental Approach in the Health and Retirement Study," *Quarterly Journal of Economics* **112**(2), 537-79.

[11] Bils, Mark, and Peter J. Klenow (2000), "Does Schooling Cause Growth?," *American Economic Review,* **90**(5), 1160–83.

[12] Blundell, Richard, Luigi Pistaferri, and Ian Preston (2008), "consumption dispersion and Partial Insurance," *American Economic Review*, **98**(5), 1887-1921.

[13] Blundell, Richard and Ian Preston (1998), "consumption dispersion and income uncertainty," *Quarterly Journal of Economics* **113**(2), 603-640.

[14] Browning, Martin, Lars Peter Hansen and James J. Heckman (1999), "Micro Data and General Equilibrium Models," Handbook of Macroeconomics.

[15] Cagetti Marco (2003), "Wealth Accumulation Over the Life Cycle and Precautionary Savings," *Journal of Business & Economic Statistics* **21**(3), 339-53.

[16] Cagetti Marco and Mariacristina De Nardi (2005), "Entrepreneurship, Frictions, and Wealth," *Journal of Political Economy* **114**(5), 835-70.

[17] Carrol, Christopher (2006), "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems," *Economics Letters* **91**(3), 312–20.

[18] Castaneda, Ana, Javier Diaz-Gimenez and Jose-Victor Rios-Rull (2003), "Accounting for Earnings and Wealth Inequality," *Journal of Political Economy*, **111**(4), 818-57.

[19] Cochrane, John H (1991), "A Simple Test of Consumption Insurance," *Journal of Political Economy*, **99**(3), 957-76.

[20] Deaton, Angus and Christina Paxson (1994), "Intertemporal Choice and Inequality," *Journal of Political Economy* **102**(3), 384-94.

[21] Diaz-Gimenez, Javier, Vincenzo Quadrini, and Jose-Victor Rios-Rull (1997), "Dimensions of Inequality: Facts on the U.S. Distribution of Earnings, Income and Wealth," *Federal Reserve Bank of Minneapolis Quarterly Review*, 3–21.

[22] Diaz-Gimenez, Javier, Vincenzo Quadrini, and Jose-Victor Rios-Rull (2002), "Updated Facts on the U.S. Distribution of Earnings, Income and Wealth," *Federal Reserve Bank of Minneapolis Quarterly Review*, 2–35.

[23] Dubois, Pierre (2001), "Consumption Insurance with Heterogeneous Preferences. Can Sharecropping Help Complete Markets?" Manuscript, INRA Toulouse.

[24] Gourinchas, Pierre-Olivier and Johnathan Parker (2002), "Consumption Over the Life Cycle," *Econometrica* **70**(1), 47-89.

[25] Guvenen, Fatih (2007), "Learning your Earning: Are Labor Income Shocks Really Very Persistent?" *American Economic Review* **97**(3), 687-712.

[26] Guvenen, Fatih (2009), "An Empirical Investigation of Labor Income Processes," *Review of Economic Dynamics* **12**(2), 58-79.

[27] Hause, John (1980), "The Fine Structure of Earnings and the On-the-Job Training Hypothesis," *Econometrica*, **48**, 1013-29.

[28] Heathcote, Jonathan, Fabrizio Perri and Giovanni L. Violante (2010a), "Unequal We Stand: An Empirical Analysis of Economic Inequality in the United States, 1967-2006," *Review of Economic Dynamics,* forthcoming.

[29] Heathcote, Jonathan, Kjetil Storesletten and Giovanni L. Violante (2010b), "The Macroeconomic Implications of Rising Wage Inequality in the United States," *Journal of Political Economy*, **118**(4), 681-722.

[30] Heathcote, Jonathan, Kjetil Storesletten and Giovanni L. Violante (2005), "Two Views of Inequality Over the Life-Cycle," *Journal of the European Economic Association* **3**(2-3), 765-75.

[31] Heathcote, Jonathan, Kjetil Storesletten and Giovanni L. Violante (2009), "Consumption and Labor Supply with Partial Insurance: An Analytical Framework," Mimeo.

[32] Hendricks, Lutz (2007a), "How Important is Discount Rate Heterogeneity for Wealth Inequality?", *Journal of Economic Dynamics & Control* **31**(9) 3042-68.

[33] Hendricks, Lutz (2007b), "Retirement wealth and lifetime earnings", *International Economic Review*, **48**(2) 421-56.

[34] Huggett, Mark (1996), "Wealth Distribution in Life-cycle Economies," *Journal of Monetary Economics* **38**(3), 469-94.

[35] Huggett, Mark, Gustavo Ventura and Amir Yaron (2010), "Sources of Lifetime Inequality," Mimeo.

[36] Kaplan, Greg and Giovanni L. Violante (2009), "How Much Consumption Insurance Beyond Self-Insurance?," *American Economic Journal: Macroeconomics*, **2**(4), 53-87.

[37] Keane, Michael. and Kenneth Wolpin (1997), "The Career Decisions of Young Men," *Journal of Political Economy* **105**(3), 473-522.

[38] Krueger, Dirk and Fabrizio Perri (2006), "Does Income Inequality Lead to consumption dispersion? Evidenceand Theory," *Review of Economic Studies* **73**(1), 163-93.

[39] Krusell and Smith (1998), "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy* **106**(5), 867-896.

[40] Lawrance, E (1991), "Poverty and the rate of time preference: evidence from panel data," *Journal of Political Economy* **99**(1), 54–77.

[41] Lillard, Lee A. and Yoram A.Weiss (1979), "Components of Variation in Panel Earnings Data: American Scientists, 1960-70," *Econometrica*, **47**, 437-54.

[42] Low, H., C. Meghir, and L. Pistaferri (2010), "Employment Risk and Wage Risk over the Life Cycle," forthcoming, *American Economic Review.*

[43] Mace, Barbara J., (1991) "Full Insurance in the Presence of Aggregate Uncertainty," *Journal of Political Economy* **99**(3), 928-56.

[44] Mazzocco, Maurizio and Shiv Saini (2009), "Testing Efficient Risk Sharing with Heterogeneous Risk Preferences," Mimeo.

[45] Primiceri, Giorgio and Thijs van Rens (2009), "Heterogeneous Life-Cycle Profiles, Income Risk and consumption dispersion," *Journal of Monetary Economics,* forthcoming.

[46] Schulhofer-Wohl, Sam (2010), "Heterogeneity and Tests of Risk Sharing," Mimeo.

[47] Storesletten, Kjetil, Christopher Telmer and Amir Yaron (2001), "How Important Are Idiosyncratic Shocks? Evidence from Labor Supply," *American Economic Review: Papers and Proceedings* **91**(2), 413-17.

[48] Storesletten, Kjetil, Christopher Telmer and Amir Yaron (2004), "Consumption and Risk Sharing Over the Life Cycle," *Journal of Monetary Economics* **51**(3), 609-33.

[49] Sun, Gang (2010), "Insure the Uninsurable by Yourself: Accounting for Consumption Insurance in a Life-cycle Model," Mimeo.

[50] Tauchen, George (1986), "Finite State Markov-chain Approximations to Univariate and Vector Autoregressions," *Economics Letters* **20**(2), 177–81.

[51] Townsend, Robert (1994), "Risk and Insurance in Village India," *Econometrica* **62**(3) 539-91.

Table 1: Parameters in Benchmark Models

Parameter	Value
Relative Risk Aversion	$\gamma = 2$
Mortality Risk	U.S. Female Life table 1991
Population Growth	$n = 1\%$
Gross Interest Rate	$R = 1.04$
Secular Growth	$g = 1.5\%$
Average Wage Growth	U.S. Census 1990
Restricted Income Process	$\rho = 1, \sigma_\alpha^2 = 0.2105, \sigma_\eta^2 = 0.0161, \sigma_\varepsilon^2 = 0.0630$
Heterogeneous Income Process	$\rho = 0.821, \sigma_\upsilon^2 = 0.022, \sigma_\eta^2 = 0.029, \sigma_\varepsilon^2 = 0.047$
	$\sigma_\theta^2 = 0.00038$, $cov(\theta, \upsilon) = -0.002$
Pension Tax in PAYG	$\tau = 0.1325$
Borrowing Constraints	$\underline{a}_{i,t+1} = -E_t y_{t+1,i}$

Table 2: Estimation of Distribution of Discount Rate

	RIP Model			HIP Model		
SIM	$\overline{\beta}$	$\Delta\beta$	p_l	$\overline{\beta}$	$\Delta\beta$	p_l
Benchmark	0.9870	0.030	64%	0.9972	0.035	66%
No pension	0.9559	0.085	81%	0.9566	0.080	78%
No borrowing	0.9878	0.090	81%	0.9947	0.095	84%
NBC	0.9918	0.020	56%	1.0120	0.015	55%
High Wealth	1.0035	0.045	62%	1.0133	0.050	68%
Alan and Browning (2010)	0.9359	0.090	50%	0.9447	0.090	50%
Hendricks (2007a)	0.9873	0.035	61%	0.9955	0.035	61%
CM	-	-	-	1.0170	0.017	55%

Table 3: Consumption Insurance

BPP:	Permanent: ϕ		Transitory: ψ	
Model	Instrument	OLS	Instrument	OLS
Data	BPP: 0.642		BPP: 0.053	
SIM: RIP				
$\sigma_\beta > 0$	0.87	0.84	0.098	0.098
$\sigma_\beta = 0$	0.79	0.78	0.057	0.057
SIM: HIP				
$\sigma_\beta > 0$	0.44	0.41	0.146	0.120
$\sigma_\beta = 0$	0.34	0.33	0.101	0.089
CM				
$\rho_{\beta,\theta}\sigma_\beta = 0.0034$	0.06	0.000	0.000	0.000
$\rho_{\beta,\theta}\sigma_\beta = 0.035$	0.64	0.000	0.000	0.000

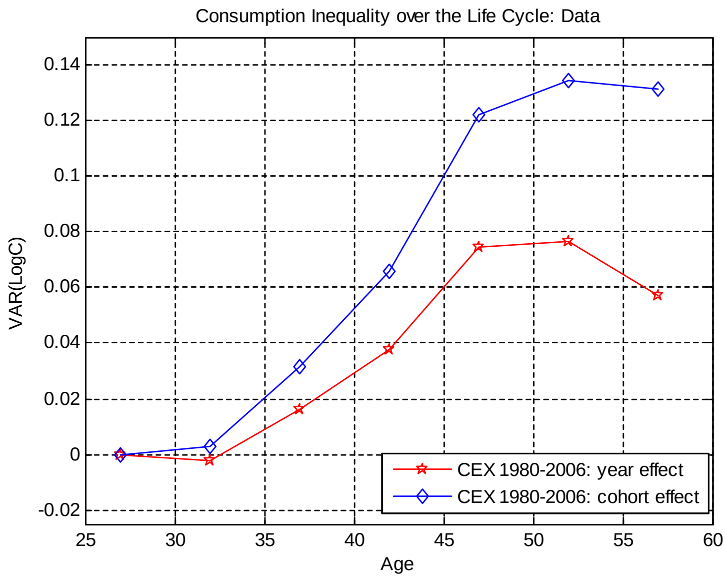

Figure 1: Consumption Inequality over the Life Cycle: Data

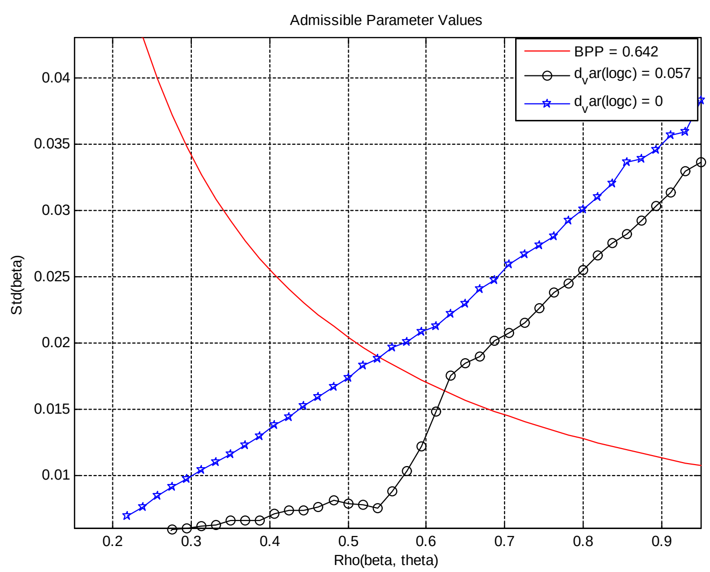

Figure 2: Matching the data: admissible pairs of $\rho_{\beta\theta}$ and σ_β

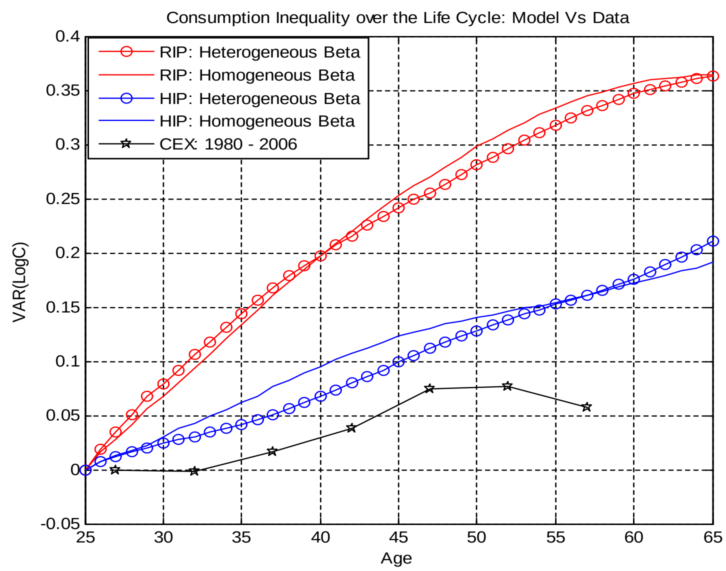

Figure 3: Consumption Inequality over the Life Cycle: SIM Models

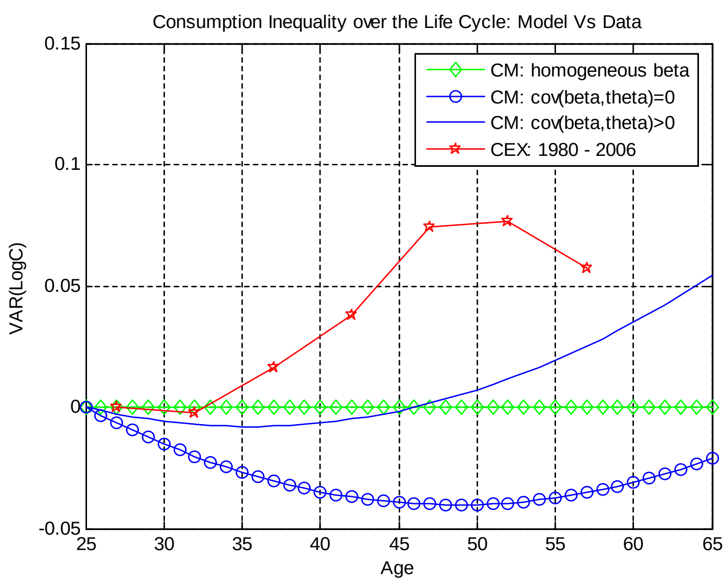

Figure 4: Consumption Inequality over the Life Cycle: CM Models

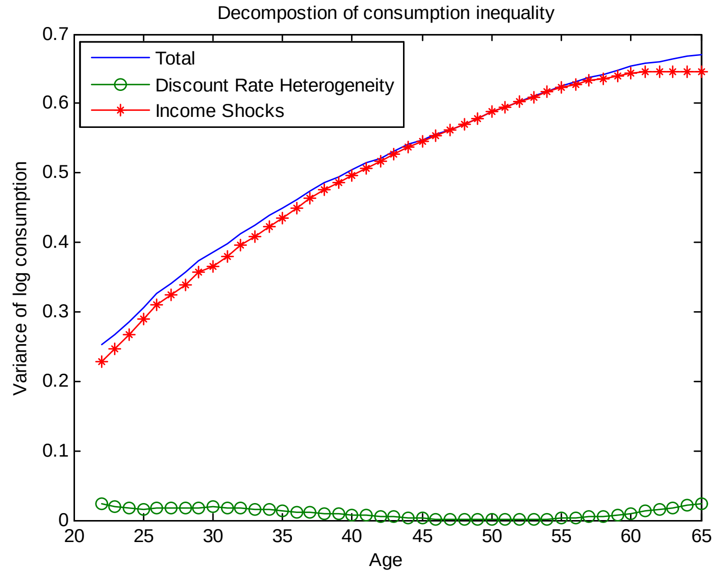

Figure 5: Decomposition of Consumption Inequality: RIP

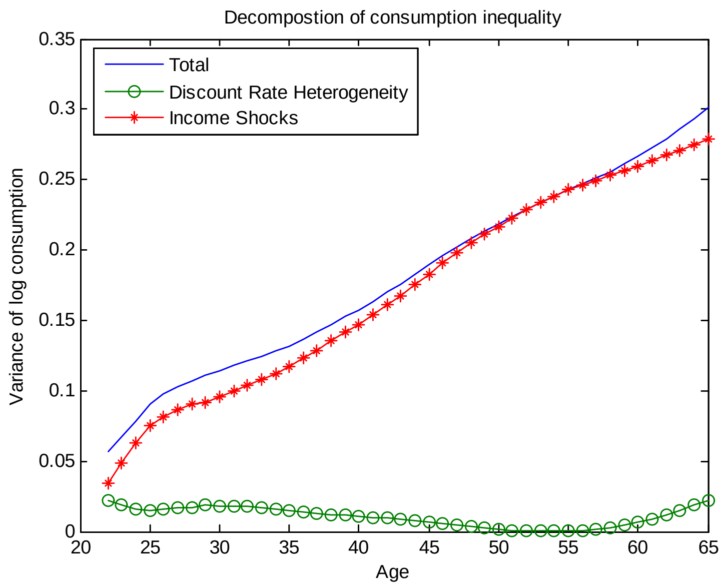

Figure 6: Decomposition of Consumption Inequality: HIP

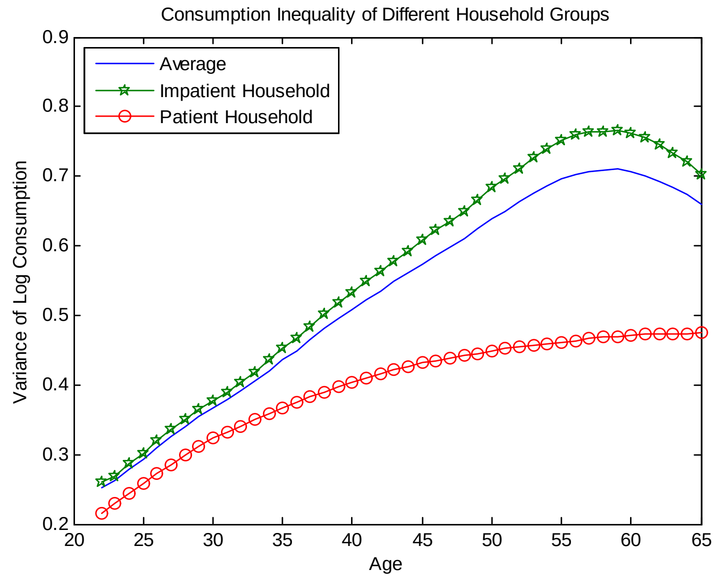

Figure 7: Consumption Inequality for Different Household Groups : RIP

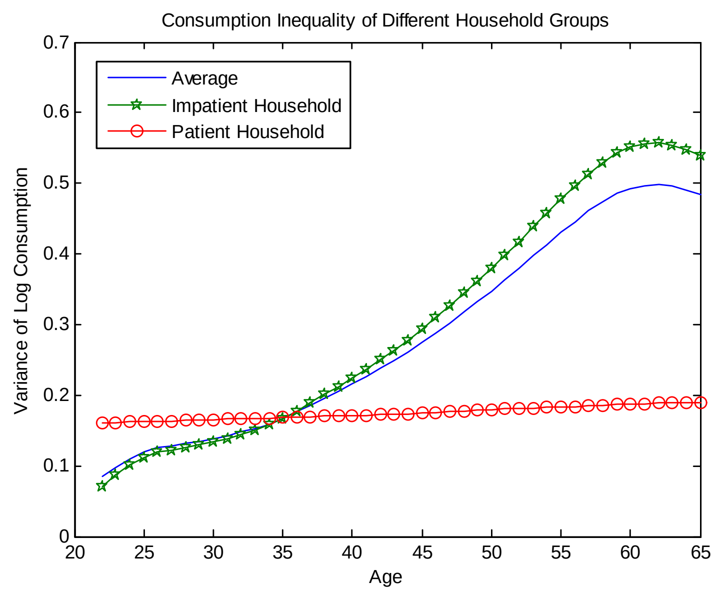

Figure 8: Consumption Inequality for Different Household Groups: HIP

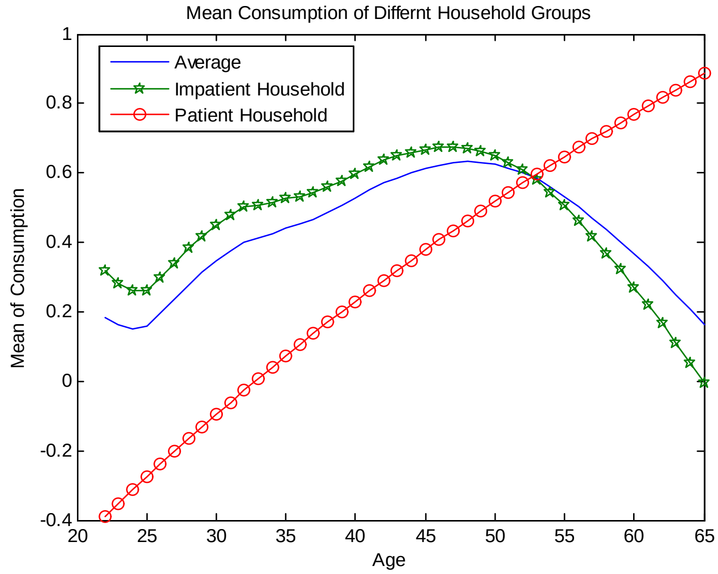

Figure 9: (Log) Consumption over the Life Cycle: RIP

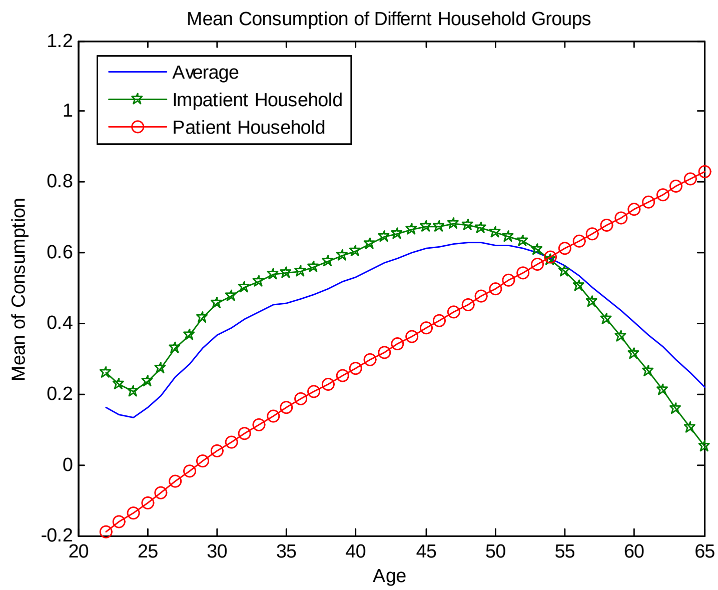

Figure 10: (Log) Consumption over the Life Cycle: HIP

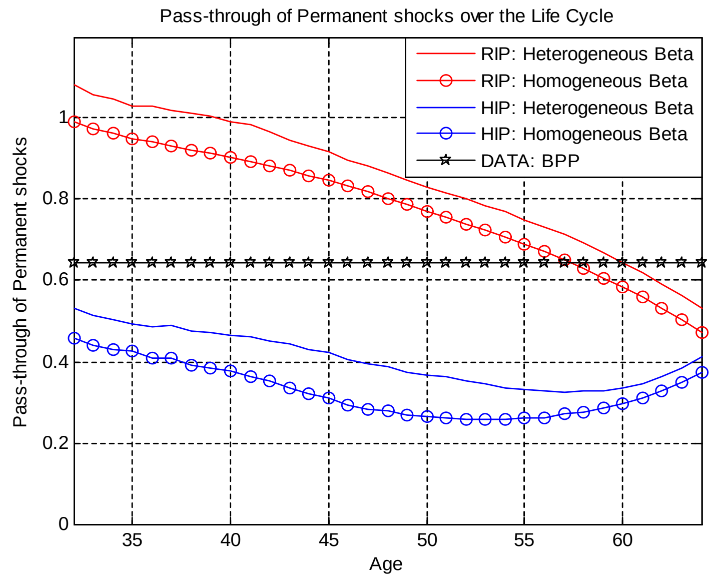

Figure 11: Pass-through of Permanent Shocks: SIM

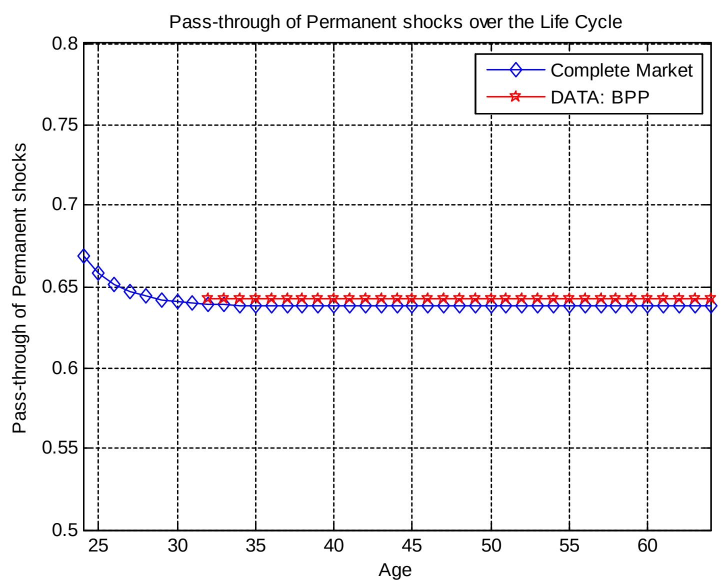

Figure 12: Pass-through of Permanent Shocks: CM

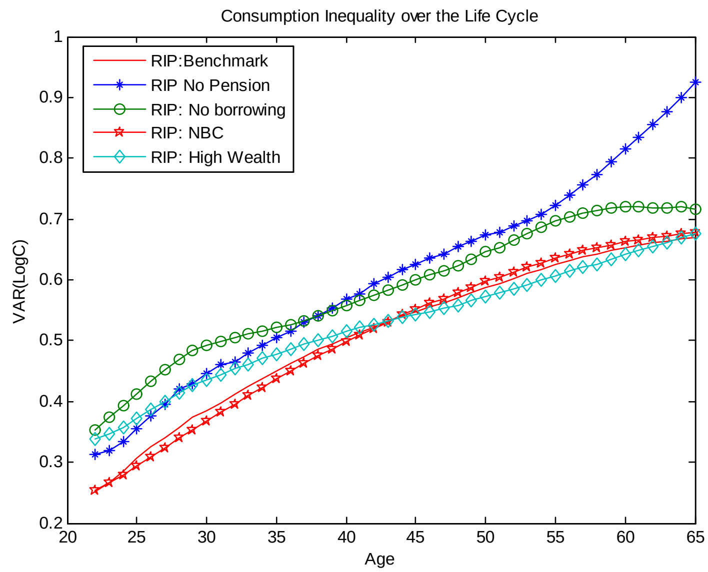

Figure 13: Age Profile of Consumption Inequality : RIP

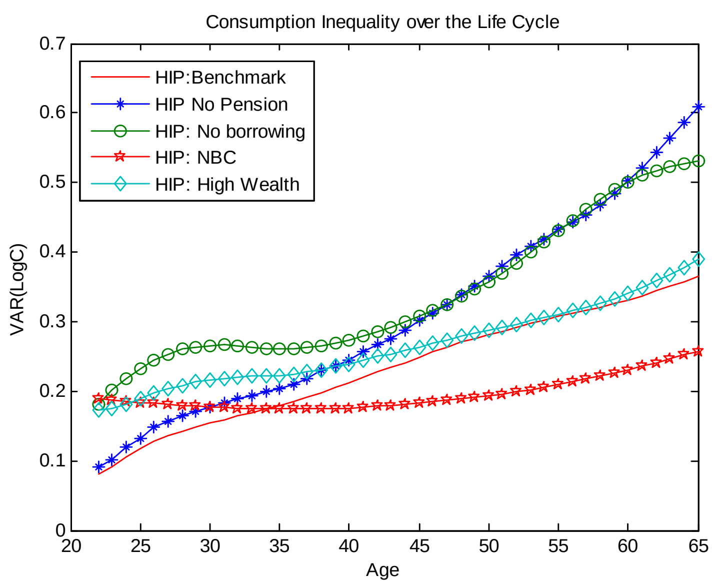

Figure 14: Age Profile of Consumption Inequality : HIP