EQUILIBRIUM WAGE/TENURE CONTRACTS WITH ON-THE-JOB LEARNING AND SEARCH.

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February, 2009

Abstract

This paper investigates equilibria in a labor market where firms post wage/tenure contracts and risk-averse workers, both employed and unemployed, search for better paid job opportunities. Different firms typically offer different contracts. Workers accumulate general human capital through on-the-job learning. With on-the-job search, a worker's wage evolves endogenously over time through experience effects, tenure effects and quits to better paid employment. This equilibrium approach suggests how to identify econometrically between experience and tenure effects on worker wages.

1 Introduction

This paper investigates individual wage dynamics in the context of an equilibrium labour market model where workers accumulate human capital while working and firms post contracts where, inter alia, wages paid depend on tenure. The analysis leads to new insights into two important areas in labour economics - the nature of equilibrium in search markets, and the empirical decomposition of wages into experience and tenure effects.

Although there is no free lunch, it is accepted by many that individuals accumulate human capital freely by working. Typists become better typists while working as typists, economists become more productive by doing economics, etc. This seems both an important and intuitive idea. A related idea now common among labour economists is that human capital can be dichotomized into general human capital and firm specific human capital. A worker who enjoys an increase in general human capital becomes more productive at all jobs, whereas accumulating firm specific human capital implies a worker is only more productive at that firm. Workers who change job, or those who are laid off, lose their firm specific human capital but keep their general human capital. Putting the above two ideas together, plus assuming a worker's wage is an increasing function of both his/her general and specific human capital, leads to at least the rudiment of a theory of how the wages of workers evolve through their working lives.

There is a significant empirical literature which has attempted to decompose wages into experience effects (general human capital) and tenure effects (firm specific capital) (see for example, Altonji and Shakotko, 1987, Topel, 1991, Altonji and Williams, 2005, and Dustmann and Meghir, 2005). The results obtained are still debated. The difficulty faced by this literature is that tenure and experience are perfectly correlated within any employment spell. As it is unreasonable to assume a quit, which resets tenure to zero, is an exogenous outcome which is orthogonal to the wage paid at the previous employer and at the new one, identifying between tenure and experience wage effects requires an equilibrium theory of wage formation and quit turnover. By extending the Burdett and Mortensen (1989) framework to allow for on-the-job learning and optimal wage tenure contracts, this paper is an important step in providing a coherent framework which properly identifies between such effects.

In this paper there is no firm specific capital by assumption. Instead wage tenure effects arise as an optimal contracting outcome when quit side-payments are ruled out by assumption. This assumption reflects an almost universal labour law - a worker can quit freely once suitable notice to leave has been given. Such legislation ensures slavery is illegal (otherwise a worker might be required to purchase his/her freedom). But in the absence of such compensatory side-payments, a firm which underpays an employee is likely to find the worker will quit to better paid employment. Of course with no search frictions, a competitive labour market automatically ensures workers earn marginal product. With search frictions, Postel-Vinay and Robin (2002), Cahuc et al (2006) find inefficient quits do not occur as firms ex-post Bertrand compete on wages to retain the contacted employee. But although such wage competition may describe a reasonable scenario in some labour markets, it may not describe all. Postel-Vinay and Robin (2004) argue instead that when on-the-job search effort is endogenous, large wage increases through Bertrand competition incentivizes employees to go out and get outside offers. Monopsonistic firms might instead simply ignore outside offers, allowing the quitter to leave and then replace the leaver through an internal promotion process. The value of this latter approach is it rewards worker loyalty to the employer. Indeed as established in Burdett and Coles (2003), a firm hierarchy where employee wages increase with tenure/seniority is an optimal wage structure which reduces quit rates and allows the firm to better exploit employee search rents.[1]

This papers extends Burdett and Coles (2003) to the case that worker productivity increases with experience; i.e. there is also on-the-job learning. This structure not only generates non-trivial, idiosyncratic earnings profiles across each worker's lifetime, it also yields equilibrium wage dispersion as:

(i) workers are ex-ante heterogeneous - worker i has productivity y_i when first entering the labour market;

[1] see Stevens (2004) for a complete discussion of optimal contracting structures when workers are risk neutral.

(ii) different employees have different total work experience x;

(iii) different employees have different tenures/seniority τ;

(iv) there is dispersion in wage contracts offered, where firm j pays piece rate $\theta = \theta_j(\tau)$ to employees with tenure τ;

(v) there is sorting with age, where on-the-job search implies workers eventually find and quit to better paid employment.

In the model considered, observed equilibrium wages can be decomposed as:

$$\log w_{ij}(x, \tau) = \log y_i + \log \theta_j(0) + \rho x + \log \frac{\theta_j(\tau)}{\theta_j(0)},$$

where ρ is the rate of human capital accumulation while employed. The observed wage thus depends on the worker fixed effect ($\log y_i$), the firm fixed effect ($\log \theta_j(0)$ which describes firm j's piece rate paid to new hires), experience effect x and the tenure effect at firm j. The identification problem is that tenure effects are firm specific and negatively correlated with the firm fixed effect. Although a standard regression equation of the form

$$\log w_{ij}(x, \tau) = \log y_i + \log \theta_j(0) + \rho x + g\tau$$

controls for worker and firm fixed effects, it is flawed as it assumes g does not vary systematically across firms. The estimated "average" tenure effect, g, may thus be small even though wage tenure effects may be large for specific firms (e.g. Dustmann and Meghir (2005)).

In the equilibrium identified below, we show the least generous firms, those which offer the lowest value contracts in the market, offer contracts with the strongest positive tenure effect. By paying a low starting wage, the firm extracts the search rents of the new hire. But paying such a low wage implies the new hire is at risk of being quickly poached by a near competitor. By raising wages quickly with tenure, the optimal contract increases the new hire's expected value to remaining with this employer and so reduces the worker's quit rate. Indeed with heterogeneous firms but no on-the-job learning, Burdett and Coles (2008) further show that "probationary" contracts can arise. Such contracts exhibit steep wage increases over very short tenures (which, in numerical simulations, can be as a short as just a few weeks). There is no training during the probationary spell. Instead the probationary phase corresponds to a quick rent-extraction phase from new hires. The insight then is that tenure effects can be very large but perhaps only at relatively short tenures and not for all firms.

A second useful insight obtained here is that optimal contracting implies wage changes within the employment spell are not directly related to the continued accumulation of experience. Instead an employment contract optimally smooths the worker's income profile against the increased risk that the worker is poached by a near competitor. Identifying experience and tenure wage effects within an employment spell is correspondingly complex.

Nevertheless an important concept is the idea of a baseline piece rate scale. As in B/C, all optimal contracts correspond to a starting point on an underlying baseline piece rate scale, where different firms offer different starting points on that scale. Over time an employee simply rises up this scale through accumulated tenure, but quits whenever an outside offer is received which puts him/her on an even higher point. This not only implies quit rates fall with tenure but also firms that pay the highest wage rates enjoy the lowest quit rates, both of which are well established empirical facts. But to formally identify tenure and experience wage effects, a seemingly promising empirical approach is to adapt the identification strategy used in Dustmann and Meghir (2005). That paper identifies experience effects by focussing on workers who are laid-off through plant closure. Given a laid-off worker i with current experience x_0, equilibrium here implies this worker's next starting wage is

$$\log w_{ij}(x,0) = \log y_i + \log \theta_j(0) + \rho x_0,$$

where $\theta_j(0)$ is a random draw from all firms making job offers in the market. Given information on experience and starting wages, this regression equation identifies ρ. Given the worker is hired at firm j, the worker's continuation wage at tenure τ is

$$\log w_{ij}(x_0+\tau,\tau) = \log y_i + \log \theta_j(0) + \rho(x_0+\tau) + \log \frac{\theta_j(\tau)}{\theta_j(0)}.$$

Thus the firm specific tenure effect at firm j is identified by

$$\log \frac{\theta_j(\tau)}{\theta_j(0)} \equiv \log \frac{w_{ij}(x_0+\tau,\tau)}{w_{ij}(x_0,0)} - \rho\tau,$$

and so identifies the piece rate scale at firm j for workers of type i. Across firms within an industry, one can then back out the industry-wide baseline piece rate scale. Furthermore the estimated firm fixed effects $\theta_j(0)$ identify the distribution of contract offers across that baseline. Identifying this structure on a matched panel data set would seem an important project for future research.

There are three closely related literatures. Bunzel et al (2000), Barlevy (2008), and Burdett et al (2008), consider the effect of on-the-job learning on wage outcomes in the Burdett and Mortensen (1998) wage posting framework. An important insight is that employment is also an investment opportunity - workers accumulate greater general human capital while employed. This implies unemployed workers have relatively low reservation wages - they are willing to "purchase" experience. Indeed unemployed worker reservation wages here are further reduced by a foot-in-the-door effect: as wages increase with seniority, an unemployed worker is willing to accept a low starting wage in order to step on the promotion ladder (also see Burdett and Coles (2003, 2008)). These effects resolve the puzzle raised in Hornstein et al (2008), that standard equilibrium wage dispersion models cannot adequately explain the difference between the lowest wage observed and the mean wage.[2] That puzzle would seemingly reflect the absence of any dynamic contracting elements in the previous search literature.

Bagger, Fontaine, Postel-Vinay and Robin (2008) instead extend the offer matching framework developed by Postel-Vinay and Robin (2002a,b) to also consider human capital accumulation with experience effects. That framework not only yields a remarkably tractable structure for econometric work, it would seem consistent with wage formation observed in many job markets, including the academic job market. Here instead we assume firms do not Bertrand compete for employees - a worker

[2] Burdett and Coles (2003, 2008) also addresses this issue through a foot-in-the-door effect: as wages increase with seniority, an unemployed worker is willing to accept a low starting wage in order to step on the promotion ladder.

simply quits if he/she receives a preferred outside offer. Of course it remains an open question whether firms behave one way or another.

Finally Shi (2008) and Menzio and Shi (2008) consider optimal wage tenure contracts but in a directed search equilibrium rather than in a random matching framework. An important advantage of that approach is that rather than consider a constant profit condition across a fixed number of firms, the number of firms is instead endogenously determined by a free entry condition. Somewhat miraculously, that framework can be extended to a business cycle framework with aggregate productivity shocks. The contract structure, where wages gradually increase through tenure, ensure employee wages do not vary much over the cycle. The issue then is whether this extended directed search approach can address the issues raised by the Shimer (2005) critique of Mortensen and Pissarides (1994).

2 THE BASIC FRAMEWORK

Time is continuous with an infinite horizon and only steady-states are considered. There is a continuum of both firms and workers, each of measure one. All firms are equally productive and have a constant returns to scale technology. Any worker's life in this market can be described by an exponential distribution with parameter $\phi > 0$. Hence, any worker leaves the market for good in any small time period dt with probability ϕdt. ϕ also describes the inflow of new labour market entrants. For ease of exposition new entrants are ex-ante identical - each entrant has the same initial productivity y_0. We shall show, however, that the results generalize straightforwardly to the case that workers are ex-ante heterogenous with initial productivity y_0 drawn from some population distribution A.

An unemployed worker's productivity y remains constant through time. On-the-job learning implies an employed worker's productivity increases at rate $\rho > 0$. Thus, after x years of work experience, a type i worker's productivity is $y = y_i e^{\rho x}$. We restrict attention to $\rho < \phi$ so that lifetime payoffs are bounded. A worker with productivity y generates flow output y while employed. We normalize the price of the production good to one, so y also describes flow revenue. Each firm pays its employees a piece rate $\theta \geq 0$. Thus an employee with productivity y who is paid piece rate θ receives flow wage $w = \theta y$. The firm enjoys corresponding profit flow $(1 - \theta)y$.

We assume the piece rate paid by a firm to a worker with productivity y depends on that worker's seniority within the firm. Specifically, each firm is characterized by a piece rate contract $\theta(\cdot)$, where a worker with tenure τ and productivity y enjoys wage $w = \theta(\tau)y$. Thus within an employment spell, the worker's wage will vary because of tenure and experience effects. We further assume anti-discrimination legislation requires that each employee is treated equally: thus each hire has to be offered the same piece-rate contract $\theta(.)$.

Workers are either unemployed or employed and obtain new job offers at Poisson rate λ, independent of their employment status. Any job offer is fully described by the contract $\theta(\cdot)$ offered by the firm; i.e., an offer is a function $\theta(t) \geq 0$ defined for all tenures $t \geq 0$. There is no recall should a worker quit or reject a job offer. Thus, given an outside contract offer, say $\widetilde{\theta}(\cdot)$, the worker compares the value of remaining at his/her current firm on contract $\theta(.)$ with current tenure τ, or switching to the new firm on contract $\widetilde{\theta}(\cdot)$ with zero tenure.

There are job destruction shocks in that each employed worker is displaced into unemployment according to a Poisson process with parameter $\delta > 0$. As typically done in this literature (e.g. Postel-Vinay and Robin (2001)), a worker with productivity y enjoys income flow by while unemployed, where $0 < b < 1$.

In the absence of job destruction shocks, workers would find their earnings always increase over time. An optimal consumption strategy with liquidity constraints would then imply workers consume current earnings $\theta(\tau)y$. Job destruction shocks, however, generate a precautionary savings motive. For tractability we simplify by assuming workers can neither borrow nor save; i.e. consumption equals earnings at all points in time. We further assume a flow utility function with constant relative risk aversion; i.e. $u(w) = w^{1-\sigma}/(1-\sigma)$ with $\sigma \geq 0$. Note that $\sigma > 1$ implies flow utility is negative and in that case we assume no suicide: there is no free exit.[3]

For simplicity assume firms and workers have a zero rate of time preference. Firms are risk neutral and, with no discounting, the objective of each firm is to maximize steady state flow profit. Each worker chooses a search and quit strategy to maximize expected total lifetime utility where the exit process implies each discounts the future at rate ϕ.

Workers

Let $V = V(y, \tau|\theta)$ denote the expected lifetime value of a worker with current productivity y and tenure τ on piece rate contract $\theta(\cdot)$. Note, that as firms always pay piece rates, wages are always proportional to productivity y. Given a CRRA utility function with parameter σ, the following identifies equilibria where V is separable with form

$$V = y^{1-\sigma} U(\tau|\theta).$$

$U(\tau|\theta)$ is referred to as the piece rate value of the contract (with tenure τ), and $U(0|\widetilde{\theta})$ is the piece rate value of the outside offer $\widetilde{\theta}$.

Let $V^U(y)$ denote the expected lifetime value of an unemployed worker with productivity y. Given a CRRA utility function, we characterize equilibria where V^U takes the form

$$V^U = y^{1-\sigma} U^U,$$

where U^U is a constant to be determined.[4]

Although firms offer piece rate contracts $\widetilde{\theta}$, workers only care about the piece rate value $U_0 = U(0|\widetilde{\theta})$ of accepting each contract offer. As firms may offer different contracts, let $F(U_0)$ denote the proportion of firms in the market whose job offer, if accepted, yields piece rate value no greater than U_0. Search is random in that given a job offer, $F(U_0)$ is the probability the offer has piece rate value no greater than U_0. Let $\underline{U}$ and $\overline{U}$ denote the infimum and supremum of this distribution function.

[3]as typically done, we assume the worker's continuation payoff is zero in the event of death but rule out endogenous exit while alive.

[4]as $U^U > 0 (< 0)$ when $\sigma < 1 (> 1)$, equilibrium payoffs are always increasing in productivity y.

Standard arguments imply the value of being an unemployed worker using an optimal search strategy satisfies the Bellman equation

$$\phi V^U = u(by) + \lambda \int_{\underline{U}}^{\overline{U}} \max[y^{1-\sigma}U_0 - V^U, 0]dF(U_0),$$

where $V^U = y^{1-\sigma}U^U$. CRRA ensures all the y terms cancel out and this Bellman equation reduces to the following equation for U^U :

$$\phi U^U = \frac{b^{1-\sigma}}{1-\sigma} + \lambda \int_{U^U}^{\overline{U}} [U_0 - U^U]dF(U_0). \tag{1}$$

Thus each unemployed worker accepts any job offer with piece rate value $U_0 \geq U^U$; i.e. CRRA implies the worker's optimal search strategy is independent of productivity y.

Now consider the value of being employed with piece rate contract $\theta(.)$. As formally established in B/C, an optimal contract implies it is never optimal for a worker to quit into unemployment.[5] Thus given an optimal contract, standard arguments imply $V = V(y, \tau|\theta(.))$ evolves according to

$$\phi V(y,\tau|\theta(.)) = u(\theta(\tau)y) + \frac{\partial V}{\partial y}\rho y + \frac{\partial V}{\partial \tau} + \lambda \int_{\underline{U}}^{\overline{U}} \max[y^{1-\sigma}U_0 - V, 0]dF(U_0) + \delta[V^U - V], \tag{2}$$

where $V(.) = y^{1-\sigma}U(\tau|\theta)$. An employed worker enjoys flow payoff $u(\theta(\tau)y)$ while employed at this firm, enjoys increasing value through on-the-job learning, enjoys changing value as tenure at the firm increases over time, at rate λ receives an outside offer with piece-rate value U_0 and quits whenever such an offer yields value exceeding V, and at rate δ becomes unemployed through a job destruction shock. CRRA and an equilibrium with the above functional forms imply the y terms all cancel out and this Bellman equation reduces to the following differential equation for $U = U(\tau|\theta)$:

$$[\delta + \phi - \rho(1-\sigma)]U - \frac{dU}{d\tau} = \frac{[\theta(\tau)]^{1-\sigma}}{1-\sigma} + \delta U^U + \lambda \int_U^{\overline{U}} [1 - F(U_0)]dU_0 \tag{3}$$

[5] Suppose an optimal contract implies the worker quits into unemployment at tenure $T \geq 0$. Thus at tenure T, the firm's continuation profit is zero and the worker obtains V^U. The same contract but which instead offers piece rate $\theta(t) = b$ for all tenures $t \geq T$ is strictly profit increasing - on-the-job learning implies the worker obtains an improved payoff no lower than V^U at T and, by not quitting, the firm's continuation payoff is strictly positive (as $b < 1$). This latter contract then makes greater expected profit which contradicts optimality of the original contract.

Again this preference structure ensures the worker's optimal quit strategy is independent of productivity y : the worker quits to any outside offer which has piece rate value greater than current value; i.e. when $U_0 \geq U = U(\tau|\theta)$. Thus each employee with tenure s at a firm with contract $\theta(\cdot)$ leaves at rate $\phi + \delta + \lambda[1 - F(U(s|\theta(.))]$. The probability a new hire survives to be an employee with tenure τ is then

$$\psi(\tau|\theta) = e^{-\int_0^\tau [\phi+\delta+\lambda(1-F(U(s|\theta)))]ds}. \tag{4}$$

Firms

Let $\overline{ue}$ denote the steady state unemployment rate and let $N(x)$ denote the fraction of unemployed workers who have experience no greater than x. Measure $1 - \overline{ue}$ of workers are thus employed and let $H(x, U)$ denote the proportion of employed workers who have experience no greater than x and piece rate value no greater than U. Each of these objects are determined endogenously.

Consider now a firm which posts contract $\theta(.)$ with starting piece rate value $U_0 = U(0|\theta)$. If $U_0 < U^U$ all potential employees prefer being unemployed to accepting this job offer and so such an offer yields zero profit. Suppose instead $U_0 \geq U^U$. As there is no discounting implies the firm's steady state flow profit can be written as[6]

$$\Omega(\theta) = \lambda \left[\begin{array}{c} \overline{ue} \int_{x=0}^{\infty} \left[\int_0^\infty \psi(\tau|\theta))[1 - \theta(\tau)][y_0 e^{\rho x}] e^{\rho\tau} d\tau \right] dN(x) \\ +(1 - \overline{ue}) \int_{U'=\underline{U}}^{U_0} \int_{x=0}^{\infty} \left[\int_0^\infty \psi(\tau|\theta)[1 - \theta(\tau)][y_0 e^{\rho x}] e^{\rho\tau} d\tau \right] dH(x, U') \end{array} \right].$$

The firm's steady state flow profit is composed of two terms. The first term describes the profit obtained by attracting unemployed workers, where the bracketed inside integral is the expected profit per hire given the new hire has starting productivity $y_0 e^{\rho x}$, and experience x is a random draw from $N(.)$. The second term describes the profit obtained by attracting employed workers who have piece rate values $U' < U_0$ and so accept the job offer. This condition can be re-expressed as

$$\begin{aligned} \Omega(\theta) &= \lambda y_0 \left[\int_0^\infty \psi(t|\theta)[1 - \theta(t)] e^{\rho t} dt \right] \\ &\times \left[\overline{ue} \int_{x=0}^{\infty} e^{\rho x} dN(x) + (1 - \overline{ue}) \int_{U'=\underline{U}}^{U_0} \int_{x=0}^{\infty} e^{\rho x} dH(x, U') \right]. \end{aligned}$$

[6] See Burdett and Coles (2003) for details.

To determine the contract that maximizes Ω we follow B/C and use the following two step procedure. First we identify a firm's piece rate contract which maximizes

$$\left[\int_0^\infty \psi(t|\theta(.))[1-\theta(t)]e^{\rho t}dt\right],$$

conditional on the contract yielding piece rate value U_0. Such a contract is termed an optimal contract. Assuming an optimal contract exists, let $\theta^*(.|U_0)$ denote it, where $\theta^*(\tau|U_0)$ is the piece rate paid at tenure τ, and define maximized profit per hire

$$\Pi^*(0|U_0) = \int_0^\infty \psi(t|\theta^*)[1-\theta^*(t|U_0)]e^{\rho t}dt.$$

An optimal contract thus yields steady-state flow profits

$$\Omega^*(U_0) = \lambda y_0 \Pi^*(0|U_0)\left[\overline{ue}\int_{x=0}^\infty e^{\rho x}dN(x) + (1-\overline{ue})\int_{U'=\underline{U}}^{U_0}\int_{x=0}^\infty e^{\rho x}dH(x,U')\right].$$

The firm's optimization problem then reduces to choosing a starting payoff U_0 to maximize $\Omega^*(U_0)$. Before formally defining an equilibrium, it is convenient first to determine the set of optimal contracts θ^*, indexed by starting value U_0, for any distribution of contract starting values F.

3 Optimal Piece Rate Tenure Contracts.

A useful preliminary insight is that because the arrival rate of offers is independent of a worker's employment status, an unemployed worker will always accept a contract which offers $\theta(\tau) = b$ for all t. Further, as $b < 1$ by assumption, a firm can always obtain strictly positive profit by offering this contract. Thus, without loss of generality, we assume (a) all firms make strictly positive profit; $\Omega^* > 0$, (b) $\underline{U} \geq U^U$ (as an offer $U_0 < U^U$ attracts no workers and so makes zero profit). We further simplify the exposition by assuming F has a connected support.

For any starting value $U_0 \geq U^U$, an optimal contract $\theta^*(.|U_0)$ solves the program

$$\max_{\theta(.)} \int_0^\infty \psi(t|\theta(.))e^{\rho t}[1-\theta(t)]dt \tag{5}$$

subject to (a) $\theta(.) \geq 0$, (b) $U(0|\theta(.)) = U_0$ and (c) the optimal quit strategies of workers which determine the survival probability $\psi(.|\theta)$. In what follows we assume the constraint $\theta \geq 0$ is never a binding constraint. Theorem 3 below shows a Market Equilibrium of this type always exists whenever $\sigma \geq 1$. In contrast, Stevens (2004) instead shows for $\sigma = 0$ (risk neutral workers) that an optimal contract always implies an initial phase where $\theta = 0$ binds. For ease of exposition, however, we do not consider this possibility.[7]

Given an optimal contract θ^* which yields starting value U_0, let $U^* \equiv U(\tau|\theta^*)$ denote the worker's piece rate value of employment at duration τ and note we can write U^* as $U^*(\tau|U_0)$. Similarly given an optimal contract θ^* which yields starting value U_0, let $\Pi^*(\tau|U_0)$ denote the firm's continuation profit given an employee with current tenure τ; i.e.

$$\Pi^*(\tau|U_0) = \int_\tau^\infty \frac{\psi(t|\theta^*)}{\psi(\tau|\theta^*)}[1 - \theta^*(t|U_0)]e^{\rho t}dt.$$

Theorem 1

For any $U_0 \geq \underline{U}$, an optimal contract $\theta^*(.|U_0)$ and corresponding worker and firm payoffs U^* and Π^* are solutions to the dynamical system $\{\theta, U, \Pi\}$ where
(a) θ is determined by

$$\begin{aligned} & \frac{\theta^{1-\sigma}}{1-\sigma} + \theta^{-\sigma}\left[1 - \theta + [\rho - \phi - \delta - \lambda(1 - F(U))]\Pi\right] \\ = \;& [\delta + \phi - \rho(1-\sigma)]U - \delta U^U - \lambda \int_U^{\overline{U}} [1 - F(U_0)]dU_0. \end{aligned} \tag{6}$$

(b) Π is given by

$$\Pi(t) = \int_t^\infty e^{-\int_t^s [\delta + \phi - \rho + \lambda(1 - F(U(\tau))]d\tau}(1 - \theta(s))ds, \text{ and} \tag{7}$$

(c) U evolves according to the differential equation

$$\frac{dU}{dt} = -\theta^{-\sigma}\frac{d\Pi}{dt} \tag{8}$$

with initial value $U(0) = U_0$.

Proof is in the Appendix.

[7]Note that for $\sigma > 0$, the corner constraint $\theta \geq 0$ is not binding in a Market Equilibrium if b is large enough.

The above characterization of an optimal contract is very general - it allows mass points in F and the density of F need not exist. In the equilibrium described in Theorem 3 below, however, the density of offers F' exists and F has a connected support. In that case, a more intuitive structure arises if we totally differentiate (6) and (7) with respect to t and so obtain the following differential equation system for (θ, Π, U):

$$\dot{\theta} = \frac{\lambda\left[\theta^{1-\sigma}\right]}{\sigma} F'(U)\Pi - \rho\theta \tag{9}$$

$$\dot{\Pi} = [\delta + \phi - \rho + \lambda(1 - F(U))]\Pi - (1 - \theta) \tag{10}$$

$$\dot{U} = -\theta^{-\sigma}\dot{\Pi} \tag{11}$$

(9) describes how piece rates optimally change along the optimal piece rate contract. The underlying structure is the same as in B/C (which set $\rho = 0$). When workers are risk neutral ($\sigma = 0$), as in Stevens (2004), the optimal contract is to pay $\theta = 0$ for a finite spell $\tau < T$, after which the worker is paid marginal product, $\theta = 1$. The intuition is that the firm wishes to extract the search rents of new hires as quickly as possible, noting that an employee who is paid less than marginal product is likely to be poached by a better paying competitor. In Stevens (2004) with risk neutral workers, there is no benefit to consumption smoothing and so the optimal starting wage hits the floor ($\theta^*(0) = 0$). When instead workers are strictly risk averse ($\sigma > 0$) and cannot borrow against future earnings, then large consumption variation within the employment spell significantly reduces the value of the contract to a new hire. An optimal contract smooths the rate at which wages increase with tenure where (9) describes the optimal trade-off: the rate at which pay is increased depends on the degree of risk aversion and on the marginal number of competing firms who are offering a starting contract with equivalent value $U_0 = U$.

On-the-job learning ($\rho > 0$), however, introduces a major difference vis-a-vis B/C. Within the employment spell, the new hire earns wage $\theta(\tau)e^{\rho\tau}\left[y_0 e^{\rho x_0}\right]$ at tenures $\tau \geq 0$, where x_0 describes the initial experience of the new hire. (9) implies the wage paid along the optimal contract evolves according to

$$\frac{\frac{d}{dt}[y_0 e^{\rho x_0} e^{\rho t}\theta(t)]}{y_0 e^{\rho x_0} e^{\rho t}\theta(t)} = \frac{\lambda F'(U)\Pi}{\sigma\theta^{\sigma}}. \tag{12}$$

As it is never efficient to pay a wage above marginal product, an optimal contract always implies strictly positive continuation profit Π. Thus, wages are always increasing within the employment spell, and increase strictly while the density of competing outside offers $F'(U) > 0$. As in B/C, the optimal contract yields a trade-off between lower wage variation (smoother consumption) and reducing marginal quit incentives. But with no learning by doing, B/C show the most generous contract offered in the market, $U_0 = \overline{U}$, implies a constant wage (perfect consumption smoothing) and the worker never quits to a competing firm. Here instead a constant wage (perfect consumption smoothing) requires a piece rate $\theta(\tau)$ which declines at rate ρ. Thus even though an optimal contract implies wages must always increase within an employment spell, tenure effects may now be negative; i.e. $\theta^*(.)$ might be a decreasing function.

As in B/C, the optimal contract is a saddle path solution to the differential equation system (9)-(11). Let $(\theta^\infty, \Pi^\infty, U^\infty)$ denote the stationary point of the dynamical system (9)-(11); i.e. $(\theta^\infty, \Pi^\infty, U^\infty)$ solves:

$$[\theta^\infty]^\sigma = \frac{\lambda}{\rho\sigma} F'(U^\infty)\Pi^\infty \tag{13}$$

$$\Pi^\infty = \frac{1-\theta^\infty}{\delta + \phi - \rho + \lambda(1 - F(U^\infty))}. \tag{14}$$

There are two types of optimal contracts, initially generous ones whose value converges to U^∞ from above, and initially ungenerous ones whose value converge to U^∞ from below. Figure 1 depicts the corresponding contracts $\theta^*(.)$ in tenure space.

Figure 1 here.

Figure 1 depicts two optimal contracts. The bottom curve depicts the optimal contract for the least generous firm, the one which offers starting payoff $U_0 = \underline{U} < U^\infty$. As in B/C the least generous contract corresponds to a path $\theta(.)$ which increases with tenure and converges to the limit value θ^∞. As piece rates increase with tenure, it follows U increases with tenure (and converges to U^∞), while continuation profit Π decreases with tenure (and converges to Π^∞). The top curve instead depicts the most generous contract which offers $U_0 = \overline{U} > U^\infty$. Although the wage paid increases with tenure, the corresponding piece rate decreases with tenure and converges to θ^∞ from above. Along that path U decreases with tenure (and converges to U^∞ from above).

It is useful to define the above contracts as the *baseline piece rate scales.* Corresponding to each tenure point on those salary scales is a unique piece rate value, which we denote $U^s(t) \in [\underline{U}, \overline{U}]$ and continuation profit $\Pi^s(t)$. Optimality of the baseline piece rate scale now yields a major simplification. Suppose a firm wishes to offer starting payoff $U_0 \in [\underline{U}, \overline{U}]$. If in addition $U_0 < U^\infty$, then optimality of the baseline piece rate scale implies the optimal contract yielding U_0 corresponds to starting point t_0 on the lower baseline piece rate scale, where $U^s(t_0) = U_0$, and piece rates paid at tenure t correspond to point $(t_0 + t)$ on the lower baseline piece rate scale. Conversely if $U_0 > U^\infty$, optimality of the baseline piece rate scale implies the optimal contract yielding U_0, is the starting point t_0 on the higher baseline piece rate scale where $U^s(t_0) = U_0$, and corresponding piece rate payments at points $(t_0 + t)$ along the higher baseline piece rate scale for all tenures $t \geq 0$.

Given this characterization of the baseline piece rate scales, we can now define and characterize a Market Equilibrium.

4 MARKET EQUILIBRIUM

A moment's reflection establishes that new hires do not care about the particular tenure contract that is offered, only the value U_0 obtained by accepting it. To proceed we transform the equations obtained above into value space (U).

Recall that for any starting value $U_0 \in [\underline{U}, \overline{U}]$, we can identify a starting point on the piece rate salary scales where the optimal contract yields starting payoff $U^s(t_0) = U_0$, yields maximized profit $\Pi^s(t_0)$ and identifies the corresponding piece rate paid θ. we can thus define $\theta = \widehat{\theta}(U_0)$ as the piece rate paid when the worker enjoys U_0 on the baseline piece rate scales, and $\Pi = \widehat{\Pi}(U_0)$ as the firm's continuation profit. Using the conditions of Theorem 1, Claim 1 identifies $\widehat{\Pi}(U)$ and $\widehat{\theta}(U)$.

Claim 1

For $U \in [\underline{U}, \overline{U}]$, $\widehat{\Pi}$ evolves according to the differential equation

$$\frac{d\widehat{\Pi}}{dU} = -\widehat{\theta}^{\sigma} \tag{15}$$

while $\widehat{\theta}$ satisfies

$$\begin{aligned} &\frac{[\widehat{\theta}]^{1-\sigma}}{1-\sigma} + \widehat{\theta}^{-\sigma}\left[1 - \widehat{\theta} + [\rho - \phi - \delta - \lambda(1 - F(U))]\widehat{\Pi}\right] \\ = \quad & [\delta + \phi - \rho(1-\sigma)]U - \delta U^U - \lambda \int_U^{\overline{U}} [1 - F(U_0)] dU_0. \end{aligned} \tag{16}$$

Proof Claim 1 follows directly from Theorem 1 and the definitions of $\widehat{\theta}, \widehat{\Pi}$.

By construction, each firm's optimized steady state flow profit by offering $U_0 \in [\underline{U}, \overline{U}]$ is

$$\Omega^*(U_0) = \lambda \widehat{\Pi}(U_0)[\overline{ue} \int_{x=0}^{\infty} y_0 e^{\rho x} dN(x) + (1 - \overline{ue}) \int_{U'=\underline{U}}^{U_0} \int_{x=0}^{\infty} y_0 e^{\rho x} dH(x, U')]. \tag{17}$$

We now formally define a Market Equilibrium.

A **Market Equilibrium** is a distribution of optimal contract offers, with corresponding value distribution $F(U)$, such that optimal job search by workers and steady state turnover implies the constant profit condition:

$$\begin{aligned} &\Omega^*(U_0) = \overline{\Omega} > 0 \text{ if } dF(U_0) > 0, \\ &\Omega^*(U_0) \leq \overline{\Omega}, \text{otherwise,} \end{aligned} \tag{18}$$

The constant profit condition requires that all optimal contracts U_0 which are offered by firms in an equilibrium must make the same profit $\overline{\Omega} > 0$, while all other contracts must make no greater profit. Below for any distribution of contract offers F, we use steady state turnover arguments to determine the equilibrium unemployment rate $\overline{ue}$ and distribution functions N, H. Identifying a Market Equilibrium then requires finding $F(.)$ so that the above constant profit condition is satisfied. we perform this task using a series of lemmas.

Lemma 1 specifies some technical results which much simplifies the exposition.

Lemma 1. A Market Equilibrium implies:

(a) $\underline{U} = U^u$;

(b) $\overline{ue} = (\phi + \delta)/(\lambda + \phi + \delta)$

(c) $F(\overline{U}) = H(\overline{U}, \infty) = 1$; i.e. there are no mass points at $U = \overline{U}$.

Proof. Lemma 1(a) implies the lowest value offer in the market equals the value of unemployment. Its proof uses simple contradiction arguments : $\underline{U} < U^u$ is inconsistent with strictly positive profit (firms offering starting value $U_0 < U^u$ make zero profit), while $\underline{U} > U^u$ is inconsistent with the constant profit condition (offering $U_0 = \underline{U}$ is dominated by offering $U_0 = U^U$ as both offers only attract the unemployed and offering $U^u < \underline{U}$ generates greater profit per hire). Lemma 1(b) follows as all unemployed workers accept their first job offer and steady state turnover implies the stated condition. Lemma 1(c) is also established with contradiction arguments. If there is a mass point in F at $\overline{U}$, then offering contract with starting value $\overline{U}$ is dominated by instead offering starting value $U_0 = \overline{U}^+$ with the following deviating contract: $\theta = \theta^{sh}(0)$ for $\tau \leq \varepsilon$, and $\theta = \theta^{sh}(\tau)$ for all $\tau > \varepsilon$, where $\varepsilon > 0$ but small. As a mass of firms offer starting contract with value $\overline{U}$, this more generous deviating contract reduces the new hire's initial quit rate by a discrete amount and so raises profit by an amount which is of order ε. As the increase in wage paid over this interval is arbitrarily small, profit per hire Π must strictly increase. As the hiring rate is no lower, this deviating contract strictly increases profit which contradicts the constant profit condition. The same contradiction argument implies H cannot contain a mass point at $\overline{U}$, otherwise the above deviating contract yields a discrete increase in the firm's hiring rate while the loss in profit per new hire is arbitrarily small. This completes the proof of Lemma 1.

The next step is to characterize steady state $N(x)$ and $H(x, U)$. The turnover arguments in Burdett et al (2008) imply the distribution of experience across unemployed workers is:

$$N(x) = 1 - \frac{\lambda\delta}{(\phi + \lambda)(\phi + \delta)} e^{-\frac{\phi(\phi+\delta+\lambda)x}{(\phi+\lambda)}}. \tag{19}$$

Let $N_0 = N(0)$ and note it is strictly positive: N_0 describes the proportion of unemployed workers who have never had a job and so have zero experience. For $x > 0$, the distribution of experience across unemployed worker is described by the exponential distribution. Burdett et al (2008) also determines the distribution of experience across all employed workers, which here is written as

$$H(x,\overline{U}) = 1 - e^{-\frac{\phi(\phi+\delta+\lambda)x}{(\phi+\lambda)}}. \tag{20}$$

In contrast to N, note that $H(0,\overline{U}) = 0$: in a steady state the measure of employed workers with zero experience must be zero. Lemma 2 now characterizes $H(.)$ for all $x > 0$, $U \in [\underline{U},\overline{U}]$.

Lemma 2. For $x > 0$ and $U \in [\underline{U},\overline{U}]$, $H = H(x,U)$ satisfies the partial differential equation:

$$[\phi + \delta + \lambda(1 - F(U)]H + \frac{\partial H}{\partial x} + \dot{U}\frac{\partial H}{\partial U} = (\phi + \delta)F(U)N(x),$$

where along the baseline piece rate scale $\dot{U} = \dot{U}(U)$ is given by:

$$\dot{U} = \widehat{\theta}^{-\sigma}[(1-\widehat{\theta}) - [\delta + \phi - \rho + \lambda(1 - F(U))]\widehat{\Pi}] \tag{21}$$

and H satisfies the boundary conditions

$$\begin{aligned} H(0,U) &= 0 \text{ for all } U \in [\underline{U},\overline{U}]; \\ H(x,\underline{U}) &= 0 \text{ for all } x \geq 0. \end{aligned}$$

The Proof of Lemma 2 is in the Appendix.

Although H is described by a relatively straightforward first order partial differential equation (pde), a closed form solution does not exist. Nevertheless it still possible to characterize fully a Market Equilibrium. Given $\overline{ue}$ obtained in Lemma 1, $N(x)$ given by (19), then (17) describing Ω^* implies the constant profit condition requires finding F such that

$$\widehat{\Pi}(U_0)\left[\begin{array}{c} \frac{\phi(\phi+\delta-\rho)}{\phi(\phi+\delta+\lambda)-\rho(\phi+\lambda)} \\ +\frac{\lambda}{\lambda+\phi+\delta}\int_{U'=\underline{U}}^{U_0}\int_{x'=0}^{\infty} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'}dx'dU' \end{array}\right] = \frac{\overline{\Omega}}{\lambda y_0} \text{ for all } U_0 \in [\underline{U},\overline{U}] \tag{22}$$

with H given by lemma 2. By solving for $\int_{U'=\underline{U}}^{U_0}\int_{x'=0}^{\infty} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'}dx'dU'$ in a Market Equilibrium, the proof of Theorem 2 shows the constant profit condition reduces to the following simple condition.

Theorem 2. In any Market Equilibrium, the constant profit condition is satisfied if and only if

$$\widehat{\Pi} = \frac{1}{\phi + \delta - \rho}\sqrt{(1-\overline{\theta})(1-\widehat{\theta})} \text{ for all } U_0 \in [\underline{U}, \overline{U}],$$

where $\overline{\theta} = \widehat{\theta}(\overline{U})$ is the highest piece rate offered in the market.

Proof of Theorem 2 is in the Appendix.

Putting $\rho = 0$ finds this solution is the same as that found in B/C. B/C showed that the implied distribution of piece rates paid then has the same functional form as the distribution of wages paid in the original Burdett and Mortensen (1998) framework. But here the distribution of wages paid also depends on ρ, the rate of on-the-job learning. Burdett et al (2008) show, with on-the-job learning and in the Burdett and Mortensen (1998) framework, that the distribution of wages paid is not only single peaked, the right tail has the Pareto distribution (i.e. has a fat right tail). Using the above solution of the constant profit condition, it is now straightforward to characterize a Market equilibrium and establish existence.

5 Existence and Characterization of a Market Equilibrium.

Although an analytic solution does not exist, solving for a Market Equilibrium is relatively straightforward. The approach is to first hypothesize an equilibrium value for $\overline{\theta}$ and then use backward induction to map out the equilibrium outcomes. The free choice of $\overline{\theta}$ is then tied down by the requirement $U^U = \underline{U}$ (lemma 1).

Conditional on an equilibrium value for $\overline{\theta}$, Lemma 3 first fully describes the equilibrium support of offers $[\underline{U}, \overline{U}]$ and obtains a closed form solution for equilibrium $\widehat{\theta}(.)$.

Lemma 3. For any equilibrium value $\overline{\theta} \in (0, 1)$, a Market Equilibrium implies $\widehat{\theta}(.)$

is given by the implicit function

$$\frac{\sqrt{(1-\overline{\theta})}}{2(\phi+\delta-\rho)}\int_{\widehat{\theta}}^{\overline{\theta}}\frac{1}{(1-\theta')^{1/2}\,[\theta']^{\sigma}}d\theta' = [\overline{U}-U] \tag{23}$$

for all $U \in [\underline{U}, \overline{U}]$ where $\overline{U}, \underline{U}$ are uniquely determined by

$$\frac{[\overline{\theta}]^{1-\sigma}}{1-\sigma} = [\phi-\rho(1-\sigma)]\overline{U} + \delta[\overline{U}-\underline{U}] \tag{24}$$

$$\frac{\sqrt{(1-\overline{\theta})}}{2(\phi+\delta-\rho)}\int_{\underline{\theta}}^{\overline{\theta}}\frac{1}{(1-\theta')^{1/2}\,[\theta']^{\sigma}}d\theta' = [\overline{U}-\underline{U}], \tag{25}$$

where $\underline{\theta} \equiv \widehat{\theta}(\underline{U})$ is the lowest piece rate paid in the market and is given by

$$(1-\underline{\theta}) = \left[\frac{\phi+\delta-\rho+\lambda}{\phi+\delta-\rho}\right]^2(1-\overline{\theta}). \tag{26}$$

Proof is in the Appendix.

As we only consider equilibria where $\theta \geq 0$ does not bind, note that $\overline{\theta} < 1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2$, which implies $\underline{\theta} < 0$, is not a relevant case. Conversely for any $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$, it is trivial to show a solution to the above equations always exists, is unique, is continuous in $\overline{\theta}$ and implies $0 < \underline{\theta} < \overline{\theta}$ and $\underline{U} < \overline{U}$.

Given the solution for equilibrium $\widehat{\theta}$ as identified by Lemma 3, Theorem 2 now uniquely determines equilibrium $\widehat{\Pi}(.)$. To determine equilibrium F, define the surplus function

$$S(U) = \int_U^{\overline{U}}[1-F(U')]dU'.$$

Noting that $U^U = \underline{U}$ in a Market Equilibrium, (16) implies equilibrium S is determined by the linear differential equation

$$\frac{[\widehat{\theta}]^{1-\sigma}}{1-\sigma} + \widehat{\theta}^{-\sigma}\left[1-\widehat{\theta}+[\rho-\phi-\delta+\lambda\frac{dS}{dU}]\widehat{\Pi}\right] = [\delta+\phi-\rho(1-\sigma)]U - \delta\underline{U} - S \tag{27}$$

for all $U \in [\underline{U}, \overline{U}]$ with initial value $S(\overline{U}) = 0$. Using the solutions above for $\widehat{\theta}, \widehat{\Pi}, \underline{U}$ and noting firms make strictly positive profit, this initial value problem uniquely determines S, and thus $F(U) \equiv dS/dU + 1$. The final step, then, is to note a Market Equilibrium must also satisfy $U^U = \underline{U}$ where U^U is given by (1).

Theorem 3. [Existence and Characterization]. The necessary and sufficient conditions for a Market Equilibrium, with $\underline{\theta} > 0$, is a $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$ with
(A) the support of offers $[\underline{U}, \overline{U}]$ (and corresponding $\underline{\theta}$) given by (24)-(26), where over that support,
(B) $\widehat{\theta}(.)$ is given by (23), $\widehat{\Pi}(.)$ is given by (22), $S(.)$ is the solution to the initial value problem (27) with $S(\overline{U}) = 0$;
and U^U, given by (1), satisfies $U^U = \underline{U}$. Furthermore such a Market Equilibrium exists for any $\sigma \geq 1$.

Proof. By construction these are necessary conditions for a Market Equilibrium. Given any such solution, then by construction all optimal contracts which offer $U_0 \in [\underline{U}, \overline{U}]$ yield the same steady state flow profit. Consider now any deviating contract. Clearly, a suboptimal contract which offers $U_0 \in [\underline{U}, \overline{U}]$ yields lower profit. Further any contract which offers value $U_0 < \underline{U}$ yields zero profit as $U^U = \underline{U}$ and all workers reject such an offer. Finally any contract which offers $U_0 > \overline{U}$ attracts no more workers than an optimal contract which offers $\overline{U}$ while the latter contract earns strictly greater profit per hire. As no deviating contracts exist which yield greater profit, a solution to the above conditions identifies a Market Equilibrium.

We now establish existence of a solution when $\sigma \geq 1$. Given an arbitrary value for $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$, let $\widetilde{F}(.|\overline{\theta})$ denote the solution for F implied by solving parts A,B of Theorem 3 above. Further define $\widetilde{U}^U(\overline{\theta})$ as the solution for U^U where

$$\phi U^U = \frac{b^{1-\sigma}}{1-\sigma} + \lambda \int_{U^U}^{\overline{U}} [1 - \widetilde{F}(.|\overline{\theta})] dU_0; \tag{28}$$

i.e. $\widetilde{U}^U(\overline{\theta})$ is the optimal reservation piece rate U^U of unemployed workers given offer distribution $\widetilde{F}(.|\overline{\theta})$. A Market Equilibrium requires finding a $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$ such that $\widetilde{U}^U(\overline{\theta}) = \underline{U}(\overline{\theta})$.

First note that as $\overline{\theta} \to 1$, (26) implies $\underline{\theta} \to 1$. As all piece rates θ paid must then lie in an arbitrarily small neighborhood around one, frictions ($\lambda < \infty$) and $b < 1$ imply $\widetilde{U}^U < \underline{U}$. Instead consider the limit $\overline{\theta} \to \lambda(\lambda + 2[\phi + \delta - \rho])/(\phi + \delta - \rho + \lambda)^2$ which, by (26), implies $\underline{\theta} \to 0$. $\sigma \geq 1$ implies the flow payoff by accepting the lowest $\underline{\theta}$ offer, $\underline{\theta}^{1-\sigma}/(1-\sigma)$, becomes unboundedly negative in this limit. As $\widehat{\theta}(.)$ is continuous and bounded above by $\overline{\theta}$, this implies $\underline{U} \to -\infty$ in this limit. But (28) implies $\phi\widetilde{U}^U > \frac{b^{1-\sigma}}{1-\sigma}$ and thus $\widetilde{U}^U > \underline{U}$ in this limit. As the solutions for $\widehat{\theta}(.), \widehat{\Pi}(.), S(.)$ and $\underline{U}$ are all continuous in $\overline{\theta}$, continuity now implies a $\overline{\theta} \in (0,1)$ exists where $\widetilde{U}^U = \underline{U}$ and so identifies a Market Equilibrium. This completes the proof of Theorem 3.

Note the existence proof does not consider the case $\sigma < 1$. If $\sigma < 1$, a Market Equilibrium might instead find the constraint $\theta \geq 0$ binds on the optimal contract (e.g. Stevens (2004) with risk neutral workers). Theorem 3 establishes this does not occur if $\sigma \geq 1$. Alternatively, as argued in B/C, one could consider $0 < \sigma < 1$ but then restrict attention to b sufficiently large that $\theta \geq 0$ does not bind.

6 Discussion and a Numerical Example.

6.1 Ex-ante worker heterogeneity.

Note that the equilibrium characterization given in Theorem 3 does not depend on y_0. It is straightforward to show the results obtained above extend directly to ex-ante heterogeneous workers, where each new entrant has productivity y_0 drawn from some population distribution A. The market outcome implied by Theorem 3 continues to describe the Market equilibrium: all workers continue to use the same turnover strategies, the unemployment rate and distribution of experience is the same for each type y_0, and the equilibrium set of optimal piece rate contracts is unchanged. The only difference is that the constant profit condition is extended to

$$\Omega^*(U_0) = \lambda\widehat{\Pi}(U_0)\int_{\underline{y}}^{\overline{y}} \left[\begin{array}{c} \overline{ue}\int_{x=0}^{\infty} y_0 e^{\rho x} dN(x) \\ +(1-\overline{ue})\int_{U'=\underline{U}}^{U_0}\int_{x=0}^{\infty} y_0 e^{\rho x} dH(x,U') \end{array} \right] dA(y_0) = \overline{\Omega} \text{ for all } U_0 \in [\underline{U}, \overline{U}] \tag{29}$$

If $\mu = \int_{\underline{y}}^{\overline{y}} y_0 dA(y_0)$, solving the constant profit condition is equivalent to

$$\widehat{\Pi}(U_0) \left[\begin{array}{c} \overline{ue}\int_{x=0}^{\infty} e^{\rho x} dN(x) \\ +(1-\overline{ue})\int_{U'=\underline{U}}^{U_0}\int_{x=0}^{\infty} e^{\rho x} dH(x,U') \end{array} \right] dA(y_0) = \frac{\overline{\Omega}}{\lambda\mu} \text{ for all } U_0 \in [\underline{U}, \overline{U}],$$

and all the results go through but with y_0 now replaced by μ.

6.2 Equilibrium Wage Equations.

If y_i denotes the initial productivity of worker i, then the equilibrium wage earned by this worker after x years experience, with tenure τ at firm j offering contract $\theta_j(.)$ is:

$$\log w_{ij}(x,\tau) = \log y_i + \log \theta_j(0) + \rho x + \log \frac{\theta_j(\tau)}{\theta_j(0)}.$$

The observed wage thus depends on the worker fixed effect ($\log y_i$), the firm fixed effect ($\log \theta_j(0)$ which describes firm j's starting piece rate paid to new hires), experience effect x and the tenure effect at firm j. Of course a Market Equilibrium not only identifies the distribution of starting piece rates $\theta_j(0)$ across all firms j, it also determines the within-firm tenure effects $\theta_j(\tau)/\theta_j(0)$. Note then the equilibrium theory finds:
(A) tenure effects are firm specific and correlated with the firm fixed effect, and also
(B) within the employment spell, optimal contracting implies wage changes are not directly related to the rate of on-the-job learning.

We first discuss point (B). Optimal contracting implies wages evolve within an employment spell according to (12); i.e.

$$\frac{d}{dt} \log w(x,\tau) = \frac{\lambda F'(U)\Pi}{\sigma \theta^{\sigma}} \tag{30}$$

and so wage growth within the employment spell does not depend directly on productivity growth ρ. This outcome reflects that an employment contract optimally smooths the worker's income profile against the increased risk that the worker is poached by a near competitor. The numerical example below finds that wage growth within the employment spell is greatest at firms at the bottom end of the market (those firms which pay starting piece rates close to $\underline{\theta}$) and only at relatively short tenures. This occurs as, by (30),
(i) along the optimal baseline piece rate scale, firms which pay low piece rates enjoy relatively high continuation profit Π per worker. Higher per worker profit implies stronger tenure effects: the firm more quickly raises wages with tenure to reduce the likelihood of a quit to a near competitor;.

(ii) an optimal contract with constant relative risk aversion implies wages rise more quickly the lower the current piece rate paid; and
(iii) a Market Equilibrium seemingly finds most equilibrium offers are concentrated around $\underline{U}$. As in B/C, there is a mass of firms which offer starting payoff $U = \underline{U}$. These firms only attract unemployed workers and so extract maximal surplus (where unemployed workers have reservation $U^U = \underline{U}$). Offering a starting payoff U_0 slightly above $\underline{U}$ is advantageous as it is likely to poach a new hire at firms in the mass point, and such hires generate a large expected profit. Of course relatively intense competition for such employees leads firms in the mass point to raise wages relatively quickly with tenure.

Point A simply reflects the construction of the baseline piece rate scale: marginal tenure effects vary systematically with the piece rate paid. Indeed Theorem 1 suggests the possibility that the highest paying firms, those that offer starting rate $\theta(0) = \overline{\theta}$, might have negative tenure effects. It turns out, however, that this does not occur in a Market Equilibrium.[8] Instead as described in B/C, a Market Equilibrium implies all equilibrium contracts exhibit positive tenure effects with limiting payoffs which converge to $(\overline{\theta}, \overline{U}, \overline{\Pi})$; i.e. there is no upper baseline piece rate scale. Nevertheless the construction of the baseline piece rate scale suggests, in equilibrium, a strong negative correlation between current piece rate earned and the marginal return to tenure.

Finally it can be shown in the frictionless limit $\lambda \to 0$, that $\overline{\Pi} \to 0$ and $\overline{\theta} \to 1$.

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[8] To establish this formally, note that differentiating (23) wrt U and then putting $U = \overline{U}$ implies

$$\frac{d\widehat{\theta}}{dU} = 2(\phi + \delta - \rho)\left[\overline{\theta}\right]^{\sigma} \tag{31}$$

which is finite. Now differentiate (16) wrt U, put $U = \overline{U}$ and simplify to get

$$\frac{d\widehat{\theta}}{dU} = \frac{\frac{\lambda\left[\overline{\theta}^{1-\sigma}\right]}{\sigma}F'(\overline{U})\overline{\Pi} - \rho\overline{\theta}}{\overline{\theta}^{-\sigma}[1 - \overline{\theta} - [\delta + \phi - \rho]\overline{\Pi}]} \tag{32}$$

at $U = \overline{U}$. (44) in the Appendix implies the denominator is zero. As $\frac{d\widehat{\theta}}{dU}$ must be finite at $\overline{U}$, this implies the numerator must also be zero; i.e. $\frac{\lambda\left[\overline{\theta}^{1-\sigma}\right]}{\sigma}F'(\overline{U})\overline{\Pi} = \rho\overline{\theta}$. This not only implies tenure effects are zero at $\overline{U}$ but also inspection finds $d\theta/dt = dU/dt = d\Pi/dt = 0$ at $\overline{U}$ and thus describes the limiting stationary point in the optimal contract problem (see Theorem 1).

Thus in the frictionless equilibrium, firms make zero profit ($\overline{\Omega} \to 0$) and all the surplus goes to the workers. Thus the extent to which firms make positive profit in a Market Equilibrium is entirely due to the presence of search frictions and the firms' collective ability to extract worker search rents.

6.3 A Numerical Example.

To illustrate the underlying tensions of market behavior, we solve the model numerically for the parameter values chosen in Burdett et al (2008). That paper considers the same matching framework except firms post piece rates rather than piece rate tenure contracts. Using a year as the reference time unit, we set $\phi = 0.025$ so that workers have a 40 year expected working lifetime. Jolivet, Postel-Vinay and Robin (2006) report turnover values $\delta = 0.08$ and $\lambda = 0.126$ for the UK.[9] we also set $\rho = 0.009$ and $b = 0.7$ which, in Burdett et al (2008), ensured a data outcome consistent with that identified in Hornstein et al (2009). we follow Lentz (2008) and set risk aversion parameter $\sigma = 2.2$. Finally for simplicity we assume all entrants are ex-ante heterogenuous with initial productivity $y_0 = 100$.

As equilibrium $\overline{\theta} \in (1-\left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$, these parameter values imply $\overline{\theta} \in (0.81, 1)$.

7 Conclusion

Becker (1974) argues in a perfectly competitive framework that wages paid should only depend on general human capital as that determines the value of the worker's outside offer. Thus in equilibrium firms pay for all firm specific capital, workers pay for all general human capital (with a lower wage to pay for any on-the-job learning) and wages do not depend on tenure. Aside from Bagger et al (2008), we are not aware of an alternative equilibrium framework where wages depend endogenously on both experience and tenure effects. A central contribution of this paper is to provide a framework within which such wage effects might be formally identified.

[9] The offer arrival rate we use is the one estimated for employed workers reported in Table 4.

This framework suggests there are two major empirical problems to identifying between experience and tenure effects on wages. First wage tenure effects are firm specific and (negatively) correlated with the firm fixed effect. Second within an employment spell, an optimal contract implies wage changes are not directly related to continued experience accumulation. The most promising approach might be to extend the identification arguments presented in Dustmann and Meghir (2005): first to identify ρ using plant closure data and then use firm level wage data to back out the underlying piece rate scale.

REFERENCES

ABOWD, J. M., F. KRAMARZ and D. N. MARGOLIS (1999), "High Wage Workers and High Wage Firms," *Econometrica*, 67, 251-334.

ABRAHAM, K.G. and H.S. FARBER (1987), "Job Duration, Seniority, and Earnings," *American Economic Review*, vol. 77, 3, 278-297.

ALBRECHT, J. and B. AXELL (1984), "An Equilibrium Model of Search Unemployment," *Journal of Political Economy* 92, 824-840.

ALTONJI, J. G. and R. A. SHAKOTKO (1987), "Do Wages Rise With Job Seniority?," *Review of Economic Studies*, LIV, 437-459.

ALTONJI, J. G. and N. WILLIAMS (1992), "The Effects of Labor Market Experience, Job Seniority and Job Mobility on Wage Growth," NBER Working Paper Series 4133.

ALTONJI, J. G. and N. WILLIAMS (1997), "Do Wages Rise With Job Seniority? A Reassessment," NBER Working Paper Series 6010.

BONTEMPS, C., J.-M. ROBIN, and G.J. VAN DEN BERG (1999), "An Empirical Equilibrium Job Search Model with Search on the Job and Heterogenous Workers," *International Economic Review* 40, 1039-1075.

BONTEMPS, C., J.-M. ROBIN, and G.J. VAN DEN BERG (2000), "Equilibrium Search with Continuous Productivity Dispersion: Theory and Non-Parameteric Estimation," *International Economic Review* 41, 305-358.

BURDETT, K and COLES, M.G. (2003), "Equilibrium Wage/Tenure Contracts," *Econometrica,* Vol. 71. No. 5 (Sept.) 1377-1404.

BURDETT, K. and MORTENSEN, D.T. (1998), "Wage Differentials, Employer Size,and Unemployment," *International Economic Review,* 39: 257-273.

CAHUC, P. F. POSTEL-VINAY,and J-M ROBIN (2006), "Wage Bargaining with on-the-job Search: Theory and Evidence", *Econometrica*, 74(2), 323-64.

DUSTMAN C. and C. MEGHIR (2005), "Wages, Experience, and Seniority," *Review of Economic Studies,* Vol. 72, No. 1, pp. 77-108, January.

ECKSTEIN, Z. and K.I. WOLPIN (1990), "Estimating a Market Equilibrium Search Model from Panel Data on Individuals," *Econometrica* 58, 783-808.

KAMOUROV, G and I. MANOVSKII (2005), "Occupation Specifitity of Human Capital," mimeo, University of Pennsylvania.

JOVANOVIC B. (1979), "Firm-specific Capital and Turnover," *Journal of Political Economy,* vol. 87, 6, 1246-1260.

JOVANOVIC B (1984), "Matching, Turnover, and Unemployment", *Journal of Political Economy*, vol. 92, 1, 108-122.

KAMBOROV G. and I MANOVSKII (2005), "Occupational Specificity of Human Capital," Working Paper, University of Pennsylvania,

LEONARD, D and N.V. LONG (1992). *Optimal Control Theory and Static Optimization in Economics,* Cambridge University Press.

MORTENSEN, D.T. (2003), *Wage Dispersion: Why are Similar People Paid Differently.* Cambridge: MIT Press, 2003.

MOSCARINI, G. (2005), "Job Matching and the Wage Distribution" *Econometrica,* Vol. 73, 481-516.

POSTEL-VINAY, F., and J.-M. ROBIN (2002a), "The Distribution of Earnings in an Equilibrium Search Model with State-Dependent Offers and Counter-Offers," *International Economic Review*, 43(4), 989-1016.

POSTEL-VINAY, F., and J.-M. ROBIN (2002b), “"Equilibrium Wage Dispersion with Worker and Employer Heterogeneity.," *Econometrica*, 70(6), 2295-350.

POSTEL-VINAY, F., and J.-M. ROBIN (2004), "Optimal wage posting with endogenous worker search intensity", *Review of Economic Dynamics,* 7 (2), 297-330.

SHIMER, R. (1996), "Contracts in a Frictional Labor Market," mimeo, MIT.

STEVENS, M. (2004), "Wage-Tenure Contracts in a Frictional Labour Market: Firms' Strategies for Recruitment and Retention", *Review of Economic Studies*, v. 71(2), pp 535-551.

TOPEL R. H. (1991), "Specific Capital, Mobility, and Wages: Wages Rise with Job Seniority," *Journal of Political Economy*," 99, 145-175.

VAN DEN BERG, G.J. (1999), "Empirical Inference with Equilibrium Search Models of the Labor Market," *Economic Journal* 109, F283-F307.

SHI, S. "Directed Search for Equilibrium Wage-Tenure Contracts", Econometrica, forthcoming

8 Appendix.

Proof of Theorem 1. Let $\Phi(t) = e^{\rho t}\psi(t|\theta(.))$. The firm's optimal contract solves

$$\max_{\theta(.)\geq 0} \int_0^\infty \Phi(t)[1-\theta(t)]dt \tag{33}$$

where on-the-job learning and optimal job search by employees implies

$$\dot{\Phi} = [\rho - \phi - \delta - \lambda(1 - F(U))]\Phi \tag{34}$$

$$\frac{dU}{dt} = [\delta + \phi - \rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda \int_U^{\overline{U}} [1 - F(U_0)]dU_0. \tag{35}$$

and starting values

$$\psi(0) = 1; U(0) = U_0. \tag{36}$$

Define the Hamiltonian

$$\begin{aligned} H &= \Phi(t)[1-\theta(t)] + \mu_\Phi[\rho - \phi - \delta - \lambda(1-F(U))]\Phi \\ &\quad + \mu_U \left[[\delta + \phi - \rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda \int_U^{\overline{U}} [1 - F(U_0)]dU_0 \right]. \end{aligned}$$

Whenever the corner constraint $\theta \geq 0$ is not binding, the Maximum Principle implies optimal $\theta(.)$ satisfies the first order condition

$$\frac{\partial H}{\partial \theta} = -\Phi(t) - \mu_U \theta(t)^{-\sigma} = 0$$

and μ_Φ, μ_U evolve according to the differential equations

$$\frac{d\mu_\Phi}{dt} = -[1-\theta(t)] - \mu_\Phi[\rho - \phi - \delta - \lambda(1 - F(U))]$$

$$\frac{d\mu_U}{dt} = -\mu_\Phi \lambda F'(U)\Phi - \mu_U \left[\delta + \phi + \lambda[1 - F(U)] - \rho(1-\sigma)\right]$$

with $\Phi(t), U$ given by the differential equations stated above. No discounting implies the additional constraint $H = 0$ (**e.g.** p298, Leonard and Long (1992)) and so we also have

$$
\begin{aligned}
0 &= \Phi(t)[1-\theta(t)] + \mu_\Phi[\rho-\phi-\delta-\lambda(1-F(U))]\Phi \\
&\quad +\mu_U\left[[\delta+\phi-\rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda\int_U^{\overline{U}}[1-F(U_0)]dU_0\right].
\end{aligned}
$$

While $\theta > 0$ along the optimal path, optimality implies $\mu_U = -\Phi/(\theta^{-\sigma})$. Substituting out μ_U in the previous expression implies

$$
\begin{aligned}
0 &= [1-\theta(t)] + \mu_\Phi[\rho-\phi-\delta-\lambda(1-F(U))] \\
&\quad -\frac{1}{\theta^{-\sigma}}\left[[\delta+\phi-\rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda\int_U^{\overline{U}}[1-F(U_0)]dU_0\right].
\end{aligned} \tag{H=0}
$$

Now integrating the linear differential equation for μ_Φ yields:

$$
\mu_\Phi(t) = \int_t^\infty e^{-\int_t^s[\delta+\phi-\rho+\lambda(1-F(U(\tau))]d\tau}(1-\theta(s))ds + A_0 e^{\int_0^t[\phi+\delta+\lambda(1-F(V(x)))-\rho]dx}
$$

where A_0 is the constant of integration. Denote the first term as the firm's continuation profit $\Pi(t)$ and note $\Pi(t)$ evolves according to (?) stated in the Theorem.

Using

$$
\mu_\Phi(t) = \Pi(t) + A_0 e^{\int_0^t[\phi+\delta+\lambda(1-F(V(x)))-\rho]dx},
$$

to substitute out $\mu_\Phi(t)$ in the $[H=0]$ condition yields

$$
\begin{aligned}
0 &= [1-\theta] + \left[\Pi(t) + A_0 e^{\int_0^t[\phi+\delta+\lambda(1-F(V(x)))-\rho]dx}\right][\rho-\phi-\delta-\lambda(1-F(U))] \\
&\quad -\frac{1}{\theta^{-\sigma}}\left[[\delta+\phi-\rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda\int_U^{\overline{U}}[1-F(U_0)]dU_0\right].
\end{aligned} \tag{37}
$$

A contradiction argument now establishes $A_0 = 0$. As Π and U are uniformly bounded (to be proved), $A_0 \neq 0$ and $\phi > \rho$ implies the second term in (37) grows exponentially as $t \to \infty$. Thus, (37) requires $\theta \to 0$ in this limit. But such a contract with $b > 0$ implies all workers quit at a finite tenure date, which contradicts optimality of the contract. Thus $A_0 = 0$.

Putting $A_0 = 0$ in (37) with some rearranging yields

$$
\begin{aligned}
&\frac{[\theta(t)]^{1-\sigma}}{1-\sigma} + \theta^{-\sigma}\left[1-\theta+[\rho-\phi-\delta-\lambda(1-F(U))]\Pi(t)\right] \\
&= [\delta+\phi-\rho(1-\sigma)]U - \delta U^U - \lambda\int_U^{\overline{U}}[1-F(U_0)]dU_0.
\end{aligned}
$$

Using this to substitute out $\frac{[\theta(t)]^{1-\sigma}}{1-\sigma}$ in (**??**) and now gives:

$$\begin{aligned} \frac{dU}{dt} &= \theta^{-\sigma}\left[1-\theta+[\rho-\phi-\delta-\lambda(1-F(U))]\Pi(t)\right] \qquad (38) \\ &= -\theta^{-\sigma}\frac{d\Pi}{dt}. \qquad (39) \end{aligned}$$

then yields (8). Connected F implies $(\theta^\infty, V^\infty, \Pi^\infty)$ is the unique stationary point of this differential equation system. This completes the proof of Theorem 1.

Proof of Lemma 2: Consider the pool of employed workers who have experience no greater than $x > 0$ and piece rate value no greater than U. Then for $U < U^\infty$, the total outflow of workers from this pool, over any instant of time $dt > 0$, is

$$(1-\overline{ue})H(x,U)[\phi+\delta+\lambda(1-F(U)]dt + (1-\overline{ue})\int_{U'=\underline{U}}^{U}\int_{x'=x-dt}^{x}\frac{\partial^2 H}{\partial U \partial x}dU'dx'$$
$$+(1-\overline{ue})\int_{U'=U^s(t-dt)}^{U^s(t)}\int_{x'=0}^{x}\frac{\partial^2 H}{\partial U \partial x}dU'dx' + O(dt^2),$$

where the first term is the number who die, lose their job or quit through receiving a job offer with value greater than U, the second is the number who exit through achieving greater experience, while the third is the number who exit through internal promotion, where $U^s(t) = U$. The inflow is $\lambda\overline{ue}F(U)N(x)dt$ which is the number of unemployed workers with experience no greater than x who receive a job offer with value no greater than U. Setting inflow equal to outflow, using the solution for $\overline{ue}$ in lemma 1 and letting $dt \to 0$ implies H satisfies:

$$H(x,U)[\phi+\delta+\lambda(1-F(U)] + \int_{U'=\underline{U}}^{U}\frac{\partial^2 H}{\partial U \partial x}dU'$$
$$+\dot{U}\int_{x'=0}^{x}\frac{\partial^2 H}{\partial U \partial x}dx' = (\phi+\delta)F(U)N(x).$$

Integrating thus yields

$$H(x,U)[\phi+\delta+\lambda(1-F(U)] + [\frac{\partial H[x,U]}{\partial x} - \frac{\partial H[x,\underline{U}]}{\partial x}]$$
$$+\dot{U}\left[\frac{\partial H[x,U]}{\partial U} - \frac{\partial H[0,U]}{\partial U}\right] = (\phi+\delta)F(U)N(x).$$

But $H(0,U) = H(x,\underline{U}) = 0$ implies $\frac{\partial H[x,\underline{U}]}{\partial x} = \frac{\partial H[0,U]}{\partial U} = 0$ which with the above equation yields the stated solution. This argument but for $U > U^{\infty}$ establishes the same differential equation. This completes the proof of Lemma 2.

Proof of Theorem 2. we begin with two preliminary facts. As $\partial H(0,U)/\partial U = 0$ (by Lemma 2), then putting $x = 0$ in the pde for H implies

$$\frac{\partial H(0,U)}{\partial x} = (\phi + \delta) N_0 F(U).$$

Also using the solution for $N(.)$ straightforward algebra establishes:

$$\int_0^{\infty} e^{\rho x'} dN(x') = \frac{\phi\,(\phi+\delta+\lambda)}{\phi+\delta}\left[\frac{\phi+\delta-\rho}{\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)}\right].$$

we now turn to solving the constant profit condition. The key is to solve for $\int_{x'=0}^{\infty}\int_{U'=\underline{U}}^{U_0} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'} dx' dU'$ with $H(.)$ as described by lemma 2. First note that as the measure of employed workers with no experience is zero, then

$$\begin{aligned}\int_{x'=0}^{\infty}\int_{U'=\underline{U}}^{U_0} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'} dx' dU' &= \int_{x'>0}\int_{U'=\underline{U}}^{U_0} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'} dx' dU' \\ &= \int_{x'>0} e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x} dx'.\end{aligned}$$

Thus (22) requires solving

$$\widehat{\Pi}(U_0)\left[\frac{\phi(\phi+\delta-\rho)}{\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)} + \frac{\lambda}{\lambda+\phi+\delta}\int_{x'>0} e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x} dx'\right] = \frac{\overline{\Omega}}{\lambda y_0} \text{ for all } U_0 \in [\underline{U}, \overline{U}]. \tag{40}$$

First put $U_0 = \overline{U}$ in (40) and let $\overline{\Pi} = \widehat{\Pi}(\overline{U})$. As, by (20), $H(x,\overline{U}) = 1 - e^{-\frac{\phi(\phi+\delta+\lambda)x}{(\phi+\lambda)}}$ then straightforward algebra establishes

$$\overline{\Pi}\left[\frac{\phi(\phi+\delta-\rho)+\lambda\phi}{\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)}\right] = \frac{\overline{\Omega}}{\lambda y_0}. \tag{41}$$

Now consider $U_0 \in [\underline{U}, \overline{U}]$. Using (40) and differentiating wrt U_0 implies

$$\left[\begin{array}{c} \frac{d\widehat{\Pi}}{dU}\left[\frac{\phi(\phi+\delta-\rho)}{\phi(\phi+\delta+\lambda)-\rho(\phi+\lambda)} + \frac{\lambda}{\lambda+\phi+\delta}\int_{x'>0} e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x}dx'\right] \\ +\widehat{\Pi}\left[\frac{\lambda}{\lambda+\phi+\delta}\int_{x'>0} e^{\rho x'}\frac{\partial^2 H(x',U_0)}{\partial x \partial U}dx'\right]\end{array}\right] = 0 \text{ for all } U_0 \in [\underline{U}, \overline{U}]. \tag{42}$$

To compute the integral in the second line then, for $x > 0$, partial differentiation wrt x of the pde for H, given by lemma 2, implies

$$\dot{U}\frac{\partial^2 H}{\partial x \partial U} = (\phi+\delta)FN'(x) - \left[[\phi+\delta+\lambda(1-F)]\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2}\right],$$

where $\dot{U} = \dot{U}(U)$ is given by (21). Thus

$$\int_{x'>0} e^{\rho x'}\frac{\partial^2 H(x', U_0)}{\partial x \partial U}dx' = \frac{1}{\dot{U}}\int_{x'>0} e^{\rho x'}\left[(\phi+\delta)FN'(x) - \left[[\phi+\delta+\lambda(1-F)]\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2}\right]\right]dx'. \tag{43}$$

Straightforward algebra using the solution for $N(.)$ finds

$$\int_{x'>0} e^{\rho x'}(\phi+\delta)FN'(x)dx = \frac{\phi\lambda\delta\,(\phi+\delta+\lambda)}{(\phi+\lambda)\,[\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)]}F(U)$$

The second term is computed using an appropriate integrating factor:

$$\begin{aligned}
&\int_{x'>0} e^{\rho x'}\left[[\phi+\delta+\lambda(1-F)]\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2}\right]dx' \\
=\ &\int_{x'>0} e^{[\rho-[\phi+\delta+\lambda(1-F)]]x'}\left\{e^{[\phi+\delta+\lambda(1-F)]x'}\left[[\phi+\delta+\lambda(1-F)]\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2}\right]\right\}dx' \\
=\ &\left[e^{\rho x'}\frac{\partial H(x', U_0)}{\partial x}\right]_{0^+}^{\infty} - \int_{x'>0}[\rho-[\phi+\delta+\lambda(1-F)]]e^{\rho x'}\frac{\partial H(x', U_0)}{\partial x}dx' \\
=\ &[\phi+\delta-\rho+\lambda(1-F)]\int_{x'>0} e^{\rho x'}\frac{\partial H(x', U_0)}{\partial x}dx' - \frac{\phi(\phi+\lambda+\delta)}{\phi+\lambda}F.
\end{aligned}$$

Inserting this solution into (43) now yields a closed form expression for $\int_{x'>0} e^{\rho x'}\frac{\partial^2 H(x', U_0)}{\partial x \partial U}dx'$. Using that solution to substitute out $\int_{x'>0} e^{\rho x'}\frac{\partial^2 H(x', U_0)}{\partial x \partial U}dx'$ in (42) then yields a closed form solution for $\int_{x'>0} e^{\rho x'}\frac{\partial H(x', U_0)}{\partial x}dx'$. But this expression is the same as $\int_{x'=0}^{\infty}\int_{U'=\underline{U}}^{U_0} e^{\rho x'}\frac{\partial^2 H(x', U')}{\partial x \partial U'}dx'dU'$. Substituting this closed form solution into (22) and simplifying then yields

$$\widehat{\Pi}^2 = \frac{[\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)]\,\overline{\Omega}}{\phi(\phi+\delta-\rho)[\delta+\phi-\rho+\lambda]\lambda y_0}(1-\widehat{\theta}) \text{ for all } U_0 \in [\underline{U}, \overline{U}].$$

Using (41) to substitute out $\overline{\Omega}$ yields

$$\widehat{\Pi}^2 = \frac{\overline{\Pi}}{(\phi+\delta-\rho)}(1-\widehat{\theta}) \text{ for all } U_0 \in [\underline{U}, \overline{U}].$$

Finally setting $U_0 = \overline{U}$ implies

$$\overline{\Pi} = \frac{1-\overline{\theta}}{\phi+\delta-\rho}. \tag{44}$$

and combining the last two expressions yields the Theorem. This completes the proof of Theorem 2.

Proof of Lemma 3. Using Theorem 2 to compute $d\widehat{\Pi}/dU$ and using (15) in Claim 1 implies $\widehat{\theta}(.)$ satisfies

$$\frac{\phi(\phi+\delta-\rho)+\lambda\phi}{2\phi(\phi+\delta-\rho)[\delta+\phi-\rho+\lambda]}\sqrt{(1-\overline{\theta})}(1-\widehat{\theta})^{-1/2}\widehat{\theta}^{-\sigma}\frac{d\widehat{\theta}}{dU} = 1$$

with $\widehat{\theta} = \overline{\theta}$ at $U = \overline{U}$. Integrating implies (23). Putting $U = \overline{U}$ in (16) and using (44) in the Appendix implies (24). Noting $\partial H(x, \underline{U})/\partial x = 0$, then putting $U = \underline{U}$ in (40) and using (41), (44) and Theorem 2 yields (26). Finally putting $U = \underline{U}$ in (**??**) implies (25). This completes the proof of Lemma 3.